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THE CONVERGENCE OF THE SOLUTION OF CAPUTO FRACTIONAL REACTION DIFFUSION EQUATION WITH NUMERICAL EXAMPLES

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ABSTRACT. In this work, we have established the exponential properties of the Mittag-Leffler functions $E_{q,1}(-\lambda t^q)$ and $E_{q,q}(-\lambda t^q)$, where $0 < q < 1$, and $\lambda > 0$. Further, using these results as a tool, we have proved that the solution of the linear Caputo fractional reaction diffusion equation converges. We have also presented some numerical examples.

AMS (MOS) Subject Classification. 34A08, 35C15, 35K57.

Key Words and Phrases : Caputo Fractional Derivative, Solution of Fractional Reaction Diffusion Equation, Convergence, Mittag-Leffler Function.

1. Introduction

In the past few decades, the qualitative and quantitative study of fractional dynamic system has attained great importance due to its application in various branches of science and engineering. Also from modeling point of view, solving non-linear fractional partial diiferential equation is more useful. The application of solving nonlinear fractional differential equation can be seen in image processing. See [2, 12, 13, 16] for details. In order to solve nonlinear fractional dynamic system, by any iterative methods, initially we need to solve the corresponding linear dynamic system on its interval of existence. In this work, we consider the linear Caputo fractional reaction diffusion equation with initial and boundary conditions. The representation form for the linear fractional diffusion equation has been obtained in [3]. The Caputo fractional dynamic system yields the result of integer dynamic system as a special case. In this work, we have considered the Caputo fractional derivative, of order q, where $0 < q < 1$, with respect to time. For $q = 1$, our fractional dynamic system yields the linear reaction diffusion equation with a non-homogeneous term, and with initial and boundary conditions as special case. The solution of the linear Caputo fractional

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reaction diffusion equation involves series of Mittag-Leffler functions of order q, for $0 < q < 1$. The Mittag-Leffler functions involved are the generalization of the exponential functions. In this paper, we establish the convergence of the infinite series of the Mittag-Leffler functions $E_{q,1}(-\lambda t^q)$ and $E_{q,q}(-\lambda t^q)$ for $0 < q < 1$. This is routine when $q = 1$, since the exponential properties of the exponential functions are well established. Although, there is a research monograph [4] on the Mittag-Leffler function, the exponential properties of Mittag-Leffler function has not been established. For that purpose, we have developed two auxiliary results relative to $E_{q,q}(-\lambda t^q)$ and $E_{q,q}(-\lambda t^q)$, for $0 < q < 1$, and $\lambda > 0$. This enable us to prove the convergence of the infinite series of Mittag-Leffler functions on $[0, \infty)$ when the initial condition, the non-homogeneous term and the boundary conditions are bounded on its domain. This result has been established in our main result. In addition, in our basic numerical result section, we have presented two examples and the graphs of the corresponding solution. Our future aim is to develop a numerical code to compute the solution of the Caputo linear fractional reaction diffusion equation.

2. **Preliminary Results**

In this section, we recall some known results and definitions that are needed for our main results.

Definition 2.1. The Caputo (left-sided) fractional derivative of $u(t)$ of order $q, n - 1 \leq q \leq n$, is given by the equation:

$$ {}^cD^q u(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} u^n(s) ds, \quad t \in [0, \infty),\ t > t_0. \tag{2.1}$$

In particular, if $q = n$, an integer, then ${}^cD^q u = u^{(n)}(x)$ and ${}^cD^q u = u^{'}(x)$ if $q = 1$.

Definition 2.2. The Riemann-Liouville fractional integral of arbitrary order q defined by

$$ D^{-q} u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s) ds, \tag{2.2}$$

where $0 < q \leq 1$.

Definition 2.3. The Riemann-Liouville (left-sided) fractional derivative of $u(t)$, when $0 < q < 1$, is defined as:

$$ D^q u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{q-1} u(s) ds, \quad t > 0. \tag{2.3}$$

Note that the Caputo integral of order q for any function is same as the Riemann-Liouville integral. Next we define the Mittag Leffler functions which are useful in computing the solution of linear fractional differential equation.

Definition 2.4. The two parameter Mittag-Leffler function is defined as

$$E_{q,r}(\lambda(t^q)) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+r)}, \tag{2.4}$$

where q, $r > 0$, and λ is a constant. Furthermore, for $r = q$, (2.4) reduces to

$$E_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+q)}. \tag{2.5}$$

If $r = 1$ in (2.4), then we have:

$$E_{q,1}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+1)}. \tag{2.6}$$

If $q = 1$, then

$$E_{1,1}(\lambda t) = e^{\lambda t}, \tag{2.7}$$

where $e^{\lambda t}$ is the usual exponential function.

The Mittag-Leffler functions are the generalization of the exponential functions. See [1, 4, 7, 14] for more details on Mittag-Leffler function.

In order to obtain the representation form for the solution of linear Caputo fractional reaction diffusion equation, we need the explicit solution of the linear Caputo fractional ordinary differential equation.

Conside the linear Caputo fractional differential equaation of the form:

$${}^cD_t^q u = \lambda u + f(t), \; u(t_0) = u_0, \quad \text{where } 0 < q \leq 1. \tag{2.8}$$

The explicit solution of (2.8) is given by

$$u(t) = u_0 E_{q,1}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) f(s) ds, \tag{2.9}$$

where $E_{q,1}(-\lambda t^q)$ and $E_{q,q}(-\lambda t^q)$ are Mittag-Leffler functions. See [1, 7, 14] for more details.

Consider the Caputo fractional reaction diffusion equation:

$${}^cD_t^q u - k u_{xx} = Q(x,t), \quad (t,x) \in Q_T, \tag{2.10}$$

$$u(x,0) = f(x), \quad x \,\epsilon\, \overline{\Omega}, \tag{2.11}$$

$$u(0,t) = A(t), u(L,t) = B(t), \quad (t,x) \,\epsilon\, \Gamma_T, \tag{2.12}$$

where $\Omega = [0, L]$, $J = (0, \infty)$, $Q_T = J \times \Omega$, $k > 0$ and $\Gamma_T = (0, \infty) \times \partial\Omega$, where $\partial\Omega$ is the boundary of Ω. We also assume that $A(t), B(t) \,\epsilon\, C^1[J, R]$, $f(x) \in C^{(2+\alpha)}$ $[\Omega, R]$, where $C^{2+\alpha}$ means that $f(x)$ is the Hölder continuous function of order $2 + \alpha$, $0 < \alpha < 1$ and $Q(x,t) \,\epsilon\, C^{1,2}[Q_T, R]$.

Under these conditions, the explicit solution of the reaction diffusion fractional equation (2.10–2.12) is given by:

$$
\begin{aligned}
u(x,t) = \sum_{n=1}^{\infty} [& \int_0^L f(x_0)\phi_n(x_0)dx_0 E_{q,1}(-k\lambda_n t^q) \\
& + \int_0^t (t-s)^{q-1} E_{q,q}(-k\lambda_n (t-s)^q [\int_0^L Q(x,s)\phi_n dx] ds \\
& + k\frac{n\pi}{L}\int_0^t (t-s)^{q-1} E - q, q(-k\lambda_n (t-s)^q A(s) ds \\
& + (-1)^{n+1} k \frac{n\pi}{L}\int_0^t (t-s)^{q-1}(Eq, q(-k\lambda_n (t-s)^q B(s) ds]\phi_n(x),
\end{aligned}
\tag{2.13}
$$

where $E_{q,1}(-\lambda t^q)$ and $E_{q,q}(-\lambda t^q)$ are Mittag-Leffler functions, $0 < q \leq 1$. This has been obtained using the eigenfunction expansion method. See [3] for details. Also, note that, here $\phi_n(x) = \sin \frac{n\pi}{L}x$. For other types of boundary conditions, $\phi_n(x)$ is computed accordingly.

3. **Auxiliary Results**

The representation formula for the solution of the linear Caputo fractional diffusion equation has been obtained in [3]. However, they have not established the convergence of the solution when the initial condition, boundary condition and the non homogeneous terms are bounded on its domain. This is well known for reaction diffusion equation of integer order. It is to be noted that the explicit solution of (2.10–2.12) involves the infinite series involving the Mittag-Leffler functions. In the special case, when $q = 1$, the integer case, the solution involved will have the series involving the exponential functions. The convergence of the infinite series involving the exponential functions are relatively easy, since the exponential properties of the exponential function are well established. However the Mittag-Leffler functions do not possess the exponential properties which is essential for the convergence of the series involving Mittag-Leffler functions. In this section, we develop some properties of the Mittag-Leffler function which will be useful in establishing the convergence of the infinite series involved in the explicit representation of the solution of the Caputo reaction diffusion equation with initial and boundary condition. In this section, initially we recall some known properties of Mittag-Leffler function.

The non-negativity of $E_{q,1}(-\lambda t^q)$ and $E_{q,q}(-\lambda t^q)$, $\lambda \in R_+$, $0 < q \leq 1$ are shown with the help of graphs, i.e. figure 1 and figure 2. In figure 1, we have graphs of $E_{q,1}(-t^q)$ and for different values of q, for $0 < q \leq 1$. In figure 2, we have graphs of $E_{q,q}(-t^q)$ for $0 < q \leq 1$. The graphs have been obtained using the MATLAB code written by Podlubny. See [8] for details. One can also draw the graphs of $E_{q,1}(-\lambda t^q)$, $E_{q,q}(-\lambda t^q)$ for fixed q and varying λ using the same code. We now use

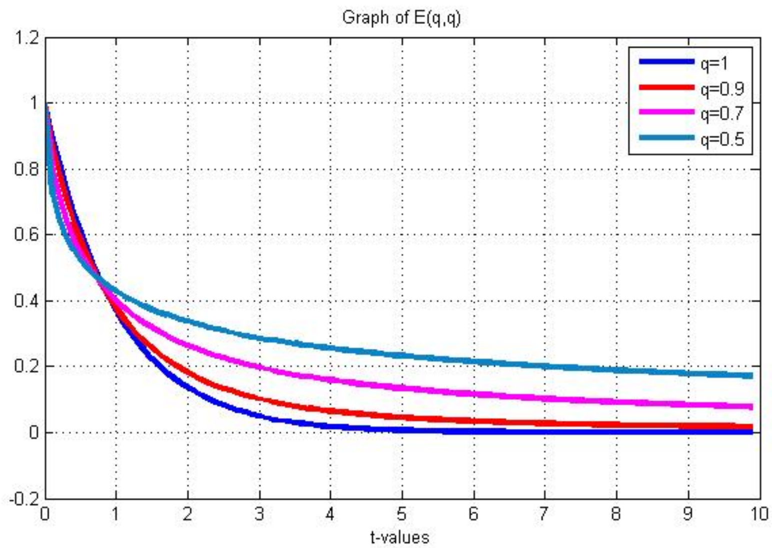

FIGURE 1. $E(q,1)$ graph
for $q = 0.5, 0.7, 0.9, 1$.

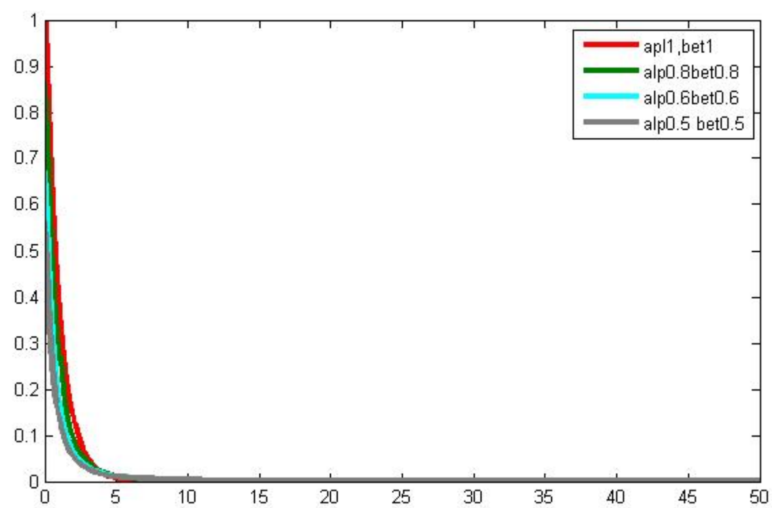

FIGURE 2. $E(q,q) graph$
for $q = 0.5, 0.6, 0.8, 1$.

the nonnegativity of the Mittag-Leffler functions to establish some kind of exponential properties of $E_{q,1}(-\lambda t^q)$ and $E_{q,q}(-\lambda t^q)$.

Lemma 3.1. *Let $E_{q,1}(-\lambda t^q)$ be the Mittag-Leffler function of order q, where $0 < q \leq 1$. Then, $\frac{E_{q,1}(-\lambda_1 t^q)}{E_{q,1}(-\lambda_2 t^q)} < 1$ where $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 = \lambda_2 + k$, for $k > 0$.*

Proof. In order to prove the result, consider

$$H(t) = E_{q,1}(-\lambda_1 t^q) - E_{q,1}(-\lambda_2 t^q). \tag{3.1}$$

Our claim is true if $H(t) < 0$. Observe that $H(0) = 0$.

Taking the Caputo derivative of order q, $0 < q < 1$, on both sides, we get

$$ {}^cD^q H(t) = -\lambda_1 E_{q,1}(-\lambda_1 t^q) + \lambda_2 E_{q,1}(-\lambda_2 t^q) \tag{3.2} $$

$$ = -\lambda_2 H(t) - k E_{q,1}(-\lambda_1 t^q). \tag{3.3} $$

Using the explicit solution of the linear fractional equation (2.9), we get

$$ H(t) = u_0 E_{q,1}(-\lambda_2 t^q) - k \int_0^t (t-s)^{q-1} E_{q,q}(-\lambda_2 (t-s)^q)) E_{q,1}(-\lambda_1 (t-s)^q) ds. \tag{3.4} $$

Since $H(0) = 0$, $k > 0$ and $E_{q,1}(-\lambda t^q)$, $E_{q,q}(-\lambda t^q)$ are positive from the graphs of figure 1 and figure 2, we have $H(t) < 0$. This proves our claim. $\square$

Lemma 3.2. *Let $E_{q,q}(-\lambda t^q)$ be the Mittag-Leffler function of order q, where $0 < q \leq 1$. Then $\frac{E_{q,q}(-\lambda_1 t^q)}{E_{q,q}(-\lambda_2 t^q)} < 1$, where $\lambda_1, \lambda_2 > 0$, such that $\lambda_1 = \lambda_2 + k$, for $k > 0$.*

Proof. The proof of this lemma follows on the same lines as that of Lemma 3.1. $\square$

4. Main Results

In this section, we recall the representation form of the solution of (2.10–2.12). It is to be noted that the solution of (2.10–2.12) can be split as $u_1(x,t)$, $u_2(x,t)$ and $u_3(x,t)$ where $u_1(x,t)$, $u_2(x,t)$ and $u_3(x,t)$ respectively are the explicit solutions of (2.10–2.12) as follows:

(a) $u_1(x,t)$ is the solution (2.13), when $Q(x,t) = 0$, $A(t) = 0 = B(t)$,
(b) $u_2(x,t)$ is the solution (2.13, when $A(t) = 0 = B(t)$, $f(x) = 0$,
(c) $u_3(x,t)$ is the solution (2.13), when $Q(x,t) = 0$, $f(x) = 0$.

Notice that $u(x,t) = u_1(x,t) + u_2(x,t) + u_3(x,t)$, where $u(x,t)$ is the solution of (2.10–2.12).

We establish individually that $u_1(x,t)$ and $u_2(x,t)$ converges for $t \in [0,\infty)$ by using results established in Lemma 3.1 and Lemma 3.2. Also, one can easily prove that $u_1(x,t)$, $u_2(x,t)$ and $u_3(x,t)$ are the unique solution when $f(x)$, $Q(x,t)$, $A(t)$ and $B(t)$ are bounded functions.

In our first main result, we establish the convergence of the solution $u_1(x,t)$ and $u_2(x,t)$ on $[0,\infty) \times [0,L]$.

Theorem 4.1. *$u_1(x,t)$ converges on $[0,\infty) \times [0,L]$ when $|f(x)| < N_1$, $N_1 > 0$, where $u_1(x,t)$ is as in (a).*

Proof. The explicit solution $u_1(x,t)$ of (2.10–2.12) is given by

$$ u_1(x,t) = \sum_{n=1}^{\infty} \Big[\int_0^L f(x_0) \phi_n(x_0) dx_0 E_{q,1}(-k\lambda_n t^q) \Big] \phi_n(x). \tag{4.1} $$

We have from (4.1)

$$|u_1(x,t)| \leq N_1 \int_o^L \sum_{n=1}^{\infty} E_{q,1}(-k\lambda_n t^q) ds,$$

since $|f(x)| < N_1$ and $|\phi_n(x)| < 1$. In the above relation, $\int_0^L ds = L$, it follows that

$$|u_1(x,t)| \leq N_1 L \sum_{n=1}^{\infty} E_{q,1}(-\lambda_n t^q).$$

Here $\lambda_n = \frac{n^2\pi^2}{L^2}$. It is enough to prove that $\sum_{n=1}^{\infty} E_{q,1}(-k\lambda_n t^q)$ converges.

Using the ratio test and Lemma 3.1, we can show that

$$\frac{E_{q,1}(-k\lambda_{n+1}t^q)}{E_{q,q}(-k\lambda_n t^q)} < 1, \quad \text{for } t > 0.$$

This proves the convergence of the series $\sum_{n=1}^{\infty} E_{q,1}(-k\lambda_n t^q)$. This concludes the proof. □

Theorem 4.2. *$u_2(x,t)$ converges on $[0,\infty) \times [0,L]$ when $|Q(x,t)| < N_2$, $N_2 > 0$, where $u_2(x,t)$ as in (b).*

Proof. The explicit solution $u_2(x,t)$ of (2.10–2.12) is given by

$$(4.2) \quad u_2(x,t) = \sum_{n=1}^{\infty} \left[\int_0^t (t-s)^{q-1} E_{q,q}(-k\lambda_n(t-s)^q)\right] \left[\int_0^L Q(x,s)\phi_n dx\right] ds]\phi_n(x).$$

We have from (4.2),

$$u_2(x,t) \leq N_2 \sum_{n=1}^{\infty} \int_0^t (t-s)^{q-1} E_{q,q}(-\lambda_n(t-s)^q) ds,$$

since $|Q(x,t)| < N_2$, $|\phi_n(x)| < 1$, the above relation becomes

$$|u_2(x,t)| \leq N_2 \sum_{n=1}^{\infty} \int_0^t (t-s)^{q-1} E_{q,q}(-\lambda_n(t-s)^q) ds.$$

That is

$$|u_2(x,t)| \leq N_2 \int_0^t \sum_{n=1}^{\infty} (t-s)^{q-1} E_{q,q}(-k\lambda_n(t-s)^q) ds.$$

Here $\lambda_n = \frac{n^2\pi^2}{L^2}$. It is enough to prove that $\sum_{n=1}^{\infty} E_{q,q}(-k\lambda_n t^q)$ converges.

Using Lemma 3.2, we have

$$\frac{E_{q,q}(-k\lambda_{n+1}(t-s)^q)}{E_{q,q}(-k\lambda_n(t-s)^q)} < 1, \quad \text{for } t > 0.$$

Using ratio test in the above inequality, we get

$$\frac{(t-s)^{q-1}E_{q,q}(-k\lambda_{n+1}(t-s)^q)}{(t-s)^{q-1}E_{q,q}(-k\lambda_n(t-s)^q)} < 1,$$

where $t > 0$. This proves the convergence of $|u_2(x,t)|$ on $[0,\infty)$. This concludes the proof. □

Theorem 4.3. *The explicit solution $u_3(x,t)$ converges on $[0,\infty)\times[0,L]$ when $|A(t)| < M_1$ and $|B(t)| < M_2$, $M_1, M_2 > 0$, where $u_3(x,t)$ as in (c).*

Proof. The proof of convergence for $u_3(x,t)$ follows on the same lines as in Theorem 4.2 except that Lemma 3.2 is used in place of Lemma 3.1. □

Hence the solution $u(x,t) = u_1(x,t)+u_2(x,t)+u_3(x,t)$ converges on $[0,\infty)\times[0,L]$, where $u(x,t)$ is the solution of (2.10–2.12).

5. Basic Numerical Results

In this section, we have provided some examples of the type $u_1(x,t)$ and $u_2(x,t)$. We have computed the solution and drawn the graphs using MATLAB and MATHEMATICA.

In example 1, we choose $Q(x,t) = A(t) = B(t) = 0$ and $f(x) = 2\sin x$. That is, we consider the linear fractional Caputo diffusion equation of the form (5.1).

Example 5.1. Consider

$$\begin{aligned} {}^cD^q u - u_{xx} &= 0, \\ u(x,0) &= 2\sin x, \\ u(0,t) &= 0 = u(1,t). \end{aligned} \tag{5.1}$$

The explicit solution of equation (5.1) is given by

$$u_1(x,t) = E_{q,1}(-\pi^2 t^q)\sin x.$$

In figure 3, we have graphed $u_1(x,t)$ as a 2D graph for given value of t and different values of q. The notation c has been used in place of t in the graph. In figure 4, we have drawn the graph of $u_1(x,t) = E_{0.7,1}(-\pi^2 t^q)\sin x$, $t \in [0,10]$.

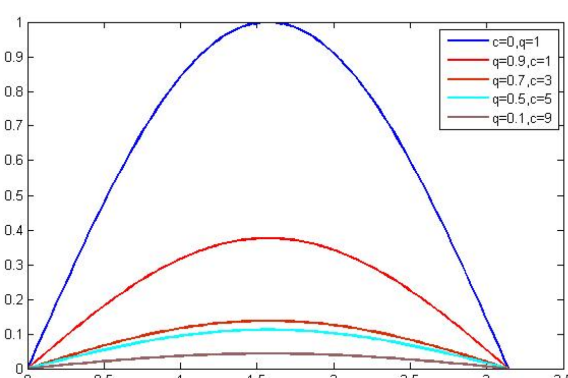

FIGURE 3. 2-d graph of $u(x,t) = \sin x * E_{q,1}(-t^q)$ at different values of q

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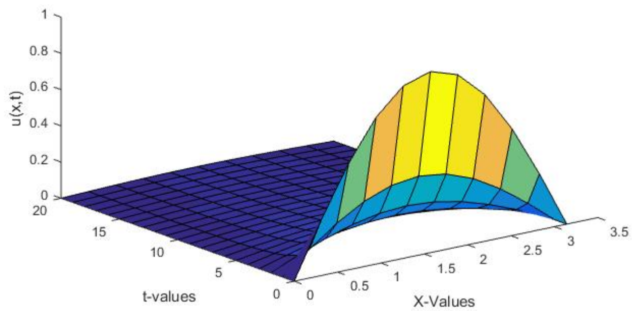

FIGURE 4. 3-d graph, graph of $u(x,t) = E_{q,1}(-\pi^2 t^q)\sin x$ when $q = 0.7$.

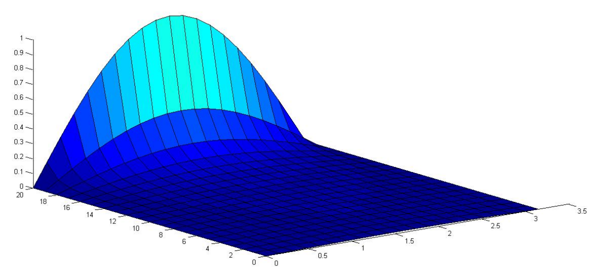

FIGURE 5. same 3-D graph as in figure 4 in another direction.

Example 5.2. Consider

$$\begin{aligned} {}^cD^q u - u_{xx} &= \Gamma(1+q)(1+t^q)\sin x; \\ u(x,0) &= 0; \\ A(t) &= 0 = B(t). \end{aligned} \tag{5.2}$$

One can easily show that the solution of (5.2) as

$$u_2(x,t) = t^q \sin x.$$

The graph of $u_2(x,t)$ is shown in figure 5 for $q = 0.9$.

6. **Concluding Remarks**

We have established the convergence of the solution of linear Caputo reaction fractional diffusion equation on the interval $[0,\infty)$. This was established using the results regarding the properties of Mittag-Leffler functions from Lemma 3.1 and Lemma 3.2. We have also presented the graphical solution of (2.10–2.12) in special cases. In the

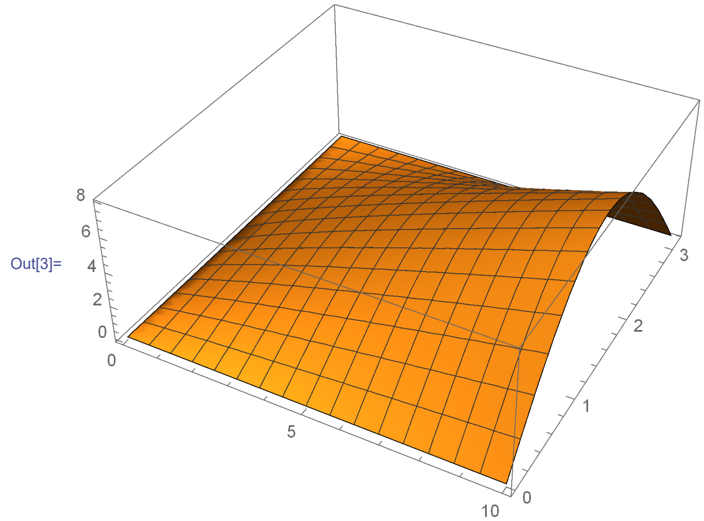

FIGURE 6. 3-d graph, graph of $u_2(x,t) = t^q \sin x$ when $q = 0.9$.

future work, we plan to develop a code to compute the solution of (2.10–2.12) numerically. This will be useful in solving nonlinear Caputo reaction fractional diffusion equation.

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