Neural, Parallel, and Scientific Computations, **26**, No. 2 (2018), 183-210 ISSN: 1061-5369

RATIONAL CUBIC FRACTAL SPLINE FOR VISUALIZATION OF SHAPED DATA

N. BALASUBRAMANI[1], M. GURU PREM PRASAD[2], AND SRINIVASAN NATESAN[3]

Department of Mathematics
[1,2,3]Indian Institute of Technology Guwahati
Guwahati, 781 039, INDIA

ABSTRACT: To interpolate the data, a new $\mathcal{C}^1$ rational fractal interpolation function (FIF) is proposed with the help of rational cubic spline which contains two families of shape parameters. The uniform error bound between the rational FIF and the original function which belongs to $\mathcal{C}^3$ is derived. The data dependent conditions on the scaling factors and on one family of the shape parameters are derived so that the constructed FIF preserves the shape features which inherited in the data.

AMS Subject Classification: 65D05, 65D07, 28A80, 41A20
Key Words: iterated function system, fractal interpolation, positivity, monotonicity, convexity

Received: October 18, 2016 ; **Accepted:** August 2, 2018 ;
Published: August 5, 2018 **doi:** 10.12732/npsc.v26i2.4
Dynamic Publishers, Inc., Acad. Publishers, Ltd. https://acadsol.eu/npsc

1. INTRODUCTION

Fractal interpolation function was introduced by Barnsley [1, 2] and it is constructed using the iteration of a finite set of contraction mappings which is called iterated function system (IFS). It generalizes the classical interpolation methods, namely polynomial interpolation and spline interpolation etc. Barnsley and Harrington [3] constructed a smooth fractal interpolation functions to approximate the unknown function. Later many authors introduced various forms of the fractal interpolation functions [5, 6, 18].

Apart from interpolating the data, sometimes it is necessary to preserve the properties or the shapes (positivity, monotonicity, convexity etc) which inherited in the data. In classical interpolation, many interpolation schemes [13, 14, 15, 20, 19, 16] were developed to preserve the shapes of the data. In case of fractal interpolation functions, Chand et al. [8, 10, 23, 12, 21, 11, 9, 22, 7] developed various interpolation schemes preserving the shapes of the data. In this paper, using rational cubic spline, a new fractal interpolation scheme that preserves the shape of the data is developed. This rational spline contains cubic polynomial in the numerator and quadratic polynomial with two shape parameters in the denominator.

The rest of the article is organized in the following way: A brief introduction about fractal interpolation function is given in Section 2. FIF with two shape parameters is constructed in Section 3 to interpolate the data that obtained from an unknown function. Section 4 deals with the uniform error bound between an original function and the FIF. Shape preserving aspects (positivity, monotonicity, convexity, constrained) of constructed fractal interpolation function are explored explicitly in Section 5. Numerical examples are provided in Section 6 to validate the theoretical results that are obtained in Section 5.

2. FRACTAL INTERPOLATION FUNCTION

Let a set of data points $\{(x_i, f_i) : i = 1, 2, \dots, N\}$ be given such that $x_1 < x_2 < \cdots < x_N$, $I = [x_1, x_N]$ and $I_i = [x_i, x_{i+1}]$. Let $L_i : I \to I_i$ be a contraction homeomorphism such that

$$L_i(x_1) = x_i, \quad L_i(x_N) = x_{i+1}.$$

for $i \in J := \{1, 2, \dots, N-1\}$. Let $K = I \times D$, where D is the compact set containing all f_i's. Consider the mappings such that for all $i \in J$, $F_i : K \to D$ satisfying

$$\begin{cases} F_i(x_1, f_1) = f_i, \quad F_i(x_N, f_N) = f_{i+1}, \\ |F_i(x, f) - F_i(x, f')| \leq |\lambda_i||f - f'|, \quad x \in I;\ f,\ f' \in D, \end{cases} \tag{2.1}$$

where $-1 < \lambda_i < 1$. For each $i \in J$, define the function $w_i : K \to K$ by $w_i(x, f) = (L_i(x), F_i(x, f))$ for all $(x, f) \in K$. The collection $\mathcal{J} = \{K; w_i : i \in J\}$ is called an IFS.

Proposition 2.1. *[2] The IFS $\{K; w_i : i \in J\}$ has a unique attractor G and G is the graph of a continuous function $f^* : I \to \mathbb{R}$ such that $f^*(x_i) = f_i$, for $i = 1, 2, \dots, N$.*

The above function f^* is called a FIF corresponding to the IFS $\mathcal{J}$ and it can also be constructed as follows:

Let $\mathcal{G} = \{g^* : I \to \mathbb{R} |\ g^* \text{ is continuous, } g^*(x_1) = f_1 \text{ and } g^*(x_N) = f_N\}$. Then $\mathcal{G}$ is a complete metric space with respect to the uniform metric $\rho(g_1^*, g_2^*) = \max\{|g_1^*(x) - g_2^*(x)| : x \in I\}$. Define the Read-Bajraktarević operator T on $(\mathcal{G}, \rho)$ as

$$Tg^*(L_i(x)) = F_i(x, g^*(x)), \quad x \in I,\ i \in J. \tag{2.2}$$

It can be seen that T is a contraction map on $(\mathcal{G}, \rho)$. Therefore, by the Banach fixed-point theorem, T has a unique fixed-point f^*(say) on $\mathcal{G}$. By (2.2), the FIF f^* satisfies the functional equation

$$f^*(L_i(x)) = F_i(x, f^*(x)), \quad x \in I,\ i \in J.$$

The FIFs constructed so far are mainly based on the IFS $\{K; w_i : i \in J\}$ with

$$\begin{cases} L_i(x) = a_i x + b_i, \\ \\ F_i(x, f) = \lambda_i f + r_i(x), \end{cases} \qquad i \in J,$$

where

$$a_i = \frac{x_{i+1} - x_i}{x_N - x_1}, \quad b_i = \frac{x_N x_i - x_1 x_{i+1}}{x_N - x_1},$$

$|\lambda_i| < 1$ and $r_i : I \to \mathbb{R}$ is a suitable continuous function such that the function F_i satisfy (2.1) for each i. The real number λ_i is called the scaling factor of the transformation w_i and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{N-1})$ is called the scale vector of the IFS. The parameter λ_i plays a crucial role in determining the shape and smoothness of the interpolant. The following proposition is established by Chand et al. to construct $\mathcal{C}^1$-rational FIFs [9].

Proposition 2.2. *Let $\{(x_i, f_i) : i = 1, 2, \ldots, N\}$ be a given data set such that $x_1 < x_2 < \cdots < x_N$. Let d_i be the derivative value at the knot x_i. Consider the IFS $\mathcal{J}^*$=$\{K;\ w_i(x, f) = (L_i(x), F_i(x, f)),\ i = 1, 2, \ldots, N-1\}$, where $L_i(x) = a_i x + b_i$, $F_i(x, f) = s_i f + r_i(x)$, $r_i(x) = p_i(x)/q_i(x)$ contains four real parameters, $p_i(x)$ is cubic polynomial, $q_i(x)$ is quadratic polynomial, $q_i(x) \neq 0$ for all $x \in [x_1, x_N]$ and $|\lambda_i| < a_i$ for all $i = 1, 2, \ldots, N-1$. Let $F_{i,1}(x, f) = (\lambda_i f + r_i^{(1)}(x))/a_i$, where $r_i^{(1)}(x)$ represents the derivative of $r_i(x)$. If for $i = 1, 2, \ldots, N-1$,*

$$F_i(x_1, f_1) = f_i, \quad F_i(x_N, f_N) = f_{i+1}, \quad F_{i,1}(x_1, d_1) = d_i, \quad F_{i,1}(x_N, d_N) = d_{i+1},$$

then the attractor of the IFS $\mathcal{J}^*$ is the graph of a $\mathcal{C}^1$-rational cubic spline FIF.

3. CONSTRUCTION OF FIF

In this section, a new rational FIF is constructed based on the rational cubic function. Let $\{(x_i, f_i) : i = 1, 2, \dots, N\}$ be a data set such that $x_1 < x_2 < \cdots < x_N$. Let d_i be the derivative value at the knot x_i. Consider the Proposition 2.2, with

$$r_i(x) = \frac{p_i(x)}{q_i(x)} \equiv \frac{P_i(\zeta)}{Q_i(\zeta)} = \frac{A_{1,i}(1-\zeta)^3 + A_{2,i}\zeta(1-\zeta)^2 + A_{3,i}\zeta^2(1-\zeta) + A_{4,i}\zeta^3}{u_i + v_i\zeta(1-\zeta)},$$

$\zeta = (x - x_1)/(x_N - x_1)$, $x \in [x_1, x_N]$. Here u_i and v_i are the shape parameters. It is assumed that $u_i > 0$ and $v_i > 0$ to avoid singularity in the denominator. Then the rational FIF satisfies the functional equation

$$\mathit{\Phi}(L_i(x)) = F_i(x, \mathit{\Phi}(x)) = \lambda_i \mathit{\Phi}(x) + r_i(x), \quad x \in I,\ i \in J. \tag{3.1}$$

The derivative $\mathit{\Phi}^{(1)}$ satisfies the following functional equation

$$\mathit{\Phi}^{(1)}(L_i(x)) = F_{i,1}(x, \mathit{\Phi}^{(1)}(x)) = \frac{\lambda_i \mathit{\Phi}^{(1)}(x) + r_i^{(1)}(x)}{a_i}, \quad x \in I,\ i \in J. \tag{3.2}$$

The constants $A_{1,i}$, $A_{2,i}$, $A_{3,i}$ and $A_{4,i}$ are evaluated based on the interpolation conditions $\mathit{\Phi}(x_i) = f_i$, $\mathit{\Phi}(x_{i+1}) = f_{i+1}$, $\mathit{\Phi}^{(1)}(x_i) = d_i$ and $\mathit{\Phi}^{(1)}(x_{i+1}) = d_{i+1}$ (these conditions are equivalent to $F_i(x_1, f_1) = f_i$, $F_i(x_N, f_N) = f_{i+1}$, $F_{i,1}(x_1, d_1)=d_i$ and $F_{i,1}(x_N, d_N)=d_{i+1}$). Let $h_i = x_{i+1} - x_i$.
Put $x = x_1$ in (3.1) to get

$$A_{1,i} = u_i[f_i - \lambda_i f_1].$$

Put $x = x_N$ in (3.1) to obtain

$$A_{4,i} = u_i[f_{i+1} - \lambda_i f_N].$$

Substituting $x = x_1$ in (3.2), we have

$$A_{2,i} = (3u_i + v_i)f_i + u_i h_i d_i - \lambda_i[(3u_i + v_i)f_1 + u_i(x_N - x_1)d_1].$$

Similarly, substituting $x = x_N$ in (3.2), we obtain

$$A_{3,i} = (3u_i + v_i)f_{i+1} - u_i h_i d_{i+1} - \lambda_i[(3u_i + v_i)f_N - u_i(x_N - x_1)d_N].$$

Thus, the rational FIF $\mathit{\Phi}$ is given by

$$\mathit{\Phi}(L_i(x)) = \lambda_i \mathit{\Phi}(x) + \frac{P_i(\zeta)}{Q_i(\zeta)}, \tag{3.3}$$

where

$$\begin{aligned} P_i(\zeta) &= (u_i[f_i - \lambda_i f_1])(1-\zeta)^3 + (u_i[f_{i+1} - \lambda_i f_N])\zeta^3 \\ &\quad + ((3u_i + v_i)f_i + u_i h_i d_i - \lambda_i[(3u_i + v_i)f_1 + u_i(x_N - x_1)d_1])\zeta(1-\zeta)^2 \\ &\quad + ((3u_i + v_i)f_{i+1} - u_i h_i d_{i+1} - \lambda_i[(3u_i + v_i)f_N - u_i(x_N - x_1)d_N])\zeta^2(1-\zeta) \\ Q_i(\zeta) &= u_i + v_i\zeta(1-\zeta),\ \zeta = (x - x_1)/(x_N - x_1),\ x \in [x_1, x_N]. \end{aligned} \tag{3.4}$$

Remark 3.1. The FIF (3.3) is the graph of the attractor of the following IFS:

$$\{K; w_i(x,f) = (L_i(x),\ F_i(x,f)),\ i = 1,2,\dots,N-1\}$$

with

$$L_i(x) = a_i x + b_i,\ F_i(x,f) = \lambda_i f + \frac{P_i(\zeta)}{Q_i(\zeta)}, \tag{3.5}$$

where a_i, b_i, $P_i(\zeta)$ and $Q_i(\zeta)$ are given as in (3.4).

Remark 3.2. If $\lambda_i = 0$ for all $i \in J$, then the rational FIF given in (3.3) reduces to the classical rational interpolant C as

$$C(x) = \frac{U_i(\eta)}{V_i(\eta)}, \tag{3.6}$$

where

$$\begin{aligned} U_i(\eta) &= u_i f_i (1-\eta)^3 + [u_i h_i d_i + f_i(3u_i + v_i)]\eta(1-\eta)^2 \\ &\quad + [-u_i h_i d_{i+1} + f_{i+1}(3u_i + v_i)]\eta^2(1-\eta) + u_i f_{i+1}\eta^3, \\ V_i(\eta) &= u_i + v_i \eta(1-\eta),\ \eta = \frac{x - x_i}{x_{i+1} - x_i},\ x \in [x_i, x_{i+1}]. \end{aligned}$$

Remark 3.3. If $\lambda_i = 0$, $u_i = 1$ and $v_i = 0$ then the FIF (3.3) reduces to the standard cubic Hermite spline.

In many situations, the derivative values d_i, $i = 1,2,\dots,N$ are not given. In such situations, the derivative values are computed by using some approximation methods. In this paper, the arithmetic mean method given in [9] is used to compute the derivative values based on the given data. These derivative values are used to construct $\mathcal{C}^1$ rational shape preserving cubic FIF.
Let $\Delta_i = (f_{i+1} - f_i)/h_i,\ i \in J$. Let $D_1^* = \Delta_1 + \dfrac{(\Delta_1 - \Delta_2)h_1}{h_1 + h_2}$, $D_i^* = \dfrac{h_i \Delta_{i-1} + h_{i-1}\Delta_i}{h_{i-1} + h_i}$, $i = 2,3,\dots,N-1$, $D_N^* = \Delta_{N-1} + \dfrac{(\Delta_{N-1} - \Delta_{N-2})h_{N-1}}{h_{N-1} + h_{N-2}}$.
At interior knots x_i, $i = 2,3,\dots,N-1$, set

$$d_i = \begin{cases} 0 & \text{if } \Delta_{i-1} = 0 \text{ or } \Delta_i = 0, \\ D_i^* & \text{otherwise},\ i = 2,3,\dots,N-1. \end{cases}$$

At end knots x_1 and x_N, set

$$d_1 = \begin{cases} 0 & \text{if } \Delta_1 = 0 \text{ or } sgn(D_1^*) \neq sgn(\Delta_1), \\ D_1^* & \text{otherwise}, \end{cases}$$

$$d_N = \begin{cases} 0 & \text{if } \Delta_{N-1} = 0 \text{ or } sgn(D_N^*) \neq sgn(\Delta_{N-1}), \\ D_N^* & \text{otherwise.} \end{cases}$$

Remark 3.4. Using the scaling factor λ_i, shape parameters u_i and v_i, shape of the curve can be modified according to the user. In particular, the scaling factor λ_i and the shape parameter v_i are playing vital role on visualizing shape of the data while u_i can take any positive value. When $\lambda_i \to 0$ and $v_i \to \infty$, the FIF (3.3) converges to a straight line in $[x_i, x_{i+1}]$. To see this, rewrite the FIF (3.3) in the following form

$$\Phi(L_i(x)) = \lambda_i \Phi(x) + \left\{ \left[(1-\zeta) f_i + \zeta f_{i+1} + \frac{R_{1i}(\zeta)}{Q_i(\zeta)} \right] - \lambda_i \left[(1-\zeta) f_1 + \zeta f_N + \frac{R_{2i}(\zeta)}{Q_i(\zeta)} \right] \right\},$$

where

$$\begin{aligned} R_{1i}(\zeta) =& u_i h_i \zeta(1-\zeta)[(\Delta_i - d_{i+1})\zeta + (d_i - \Delta_i)(1-\zeta)], \\ R_{2i}(\zeta) =& u_i \zeta(1-\zeta)[\{(f_N - f_1) - (x_N - x_1) d_N\}\zeta \\ & + \{(x_N - x_1) d_1 - (f_N - f_1)\}(1-\zeta)]. \end{aligned}$$

If $v_i \to \infty$, then Φ converges to the following affine FIF

$$\Phi(L_i(x)) = \lambda_i \Phi(x) + (f_i - \lambda_i f_1)(1-\zeta) + (f_{i+1} - \lambda_i f_N)\zeta.$$

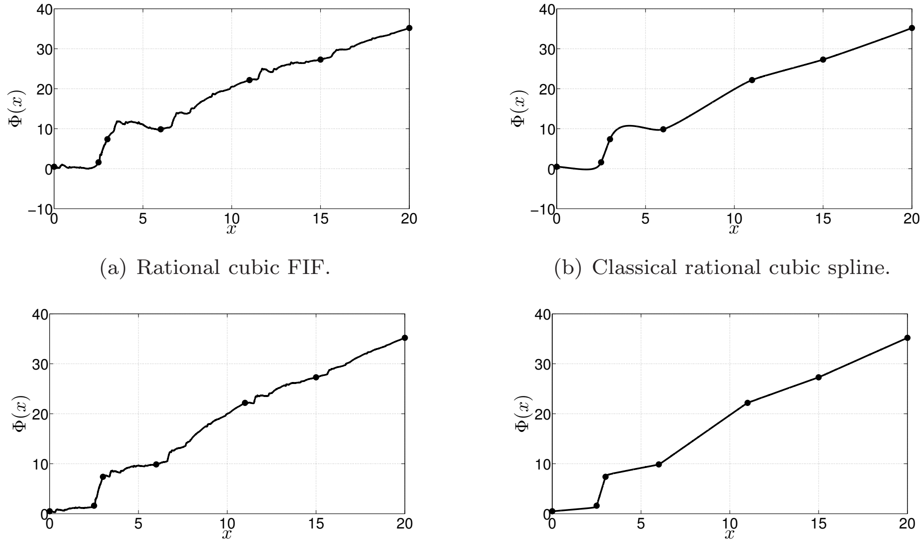
Also, if $v_i \to \infty$ and $\lambda_i \to 0$, then Φ converges to the straight line segment in the interval $[x_i, x_{i+1}]$, i.e.,

$$\Phi(L_i(x)) = f_i(1-\zeta) + f_{i+1}\zeta.$$

Example 3.5. Consider the interpolation data $\{ (0, 0.5), (2.5, 1.61), (3, 7.3891), (6, 9.8696), (11, 22.18), (15, 27.3), (20, 35.2) \}$ as used in [8]. The derivative values are approximated using arithmetic mean method and given by $d_1 = 0$, $d_2 = 9.7058$, $d_3 = 10.0251$, $d_4 = 1.4401$, $d_5 = 1.8054$, $d_6 = 1.4133$ and $d_7 = 1.7467$. Using arbitrary scaling factors and shape parameters, the rational cubic FIF is generated which is shown in Figure 1(a). Classical rational cubic spline is constructed by taking all the scaling factors $\lambda_i = 0$, $i = 1, 2, \ldots, 6$ which is shown in Figure 1(b). When $v_i \to \infty$, the rational cubic FIF becomes affine FIF which is shown in Figure 1(c). When $\lambda_i \to 0$ and $v_i \to \infty$, the FIF becomes a straight line in each interval which is shown in Figure 1(d). The parameters that are used to generate Figure 1 is given in Table 1.

4. CONVERGENCE ANALYSIS

Assume that the data $\{(x_i, f_i, d_i) : i = 1, 2, \ldots, N\}$ is generated from the function S which belongs to $\mathcal{C}^3$. In this section, the error bound between the original function



(a) Rational cubic FIF.

(b) Classical rational cubic spline.

(c) Behaviour of FIF when $v_i \to \infty$.

(d) Behaviour of FIF when $\lambda_i \to 0$ and $v_i \to \infty$.

Figure 1: Rational cubic FIF with two parameter family.

Table 1: Parameters for FIFs given in Figure 1 with $u_i = 1.5$ for $i = 1, 2, \ldots, 6$.

Figure	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	v_1	v_2	v_3	v_4	v_5	v_6
1(a)	0.1	0.12	0.2	0.28	0.24	0.13	10	9.2	3.8	5	8	3.9
1(b)	0	0	0	0	0	0	10	9.2	3.8	5	8	3.9
1(c)	0.1	0.12	0.2	0.28	0.24	0.13	90	102	98	114	128	104
1(d)	0.001	0.0013	0.003	0.0023	0.0043	0.003	90	102	98	114	128	104

S and the corresponding FIF defined in (3.3) is derived. This error bound will be estimated with the help of the corresponding classical rational cubic spline.

Theorem 4.1. *Let Φ be a rational cubic FIF given in (3.3) and C be a classical rational cubic spline given in (3.6) with respect to the data $\{(x_i, f_i, d_i) : i = 1, 2, \ldots, N\}$ which generated from an original function $S \in \mathcal{C}^3[x_1, x_N]$. Then*

$$\|S - \Phi\|_\infty \leq \frac{1}{(1 - |\lambda|_\infty)} \Big[|\lambda|_\infty \Big([M + \frac{h}{4}\overline{M}] + [M^* + \frac{|I|}{4} M_*] \Big) \Big] + \|S^{(3)}\|_\infty h^3 c^*,$$

where $M = \max\{|f_i| : i = 1, 2, \ldots, N\}$, $\overline{M} = \max\{|d_i| : i = 1, 2, \ldots, N\}$, $M^* = \max\{|f_1|, |f_N|\}$, $M_* = \max\{|d_1|, |d_N|\}$, $|I| = x_N - x_1$, $h = \max\{h_i : i \in J\}$, $|\lambda|_\infty =$

$\max\{|\lambda_i| : i \in J\}$,

$$\sigma_1(u_i, v_i, \eta) = -\frac{\eta^3}{3} + \frac{(1-\eta)^3\eta^2\{(v_i + \eta(u_i - v_i)\}}{3V_i(\eta)} + \frac{8u_i^2(1-\eta)^3\eta^2}{3V_i(\eta)[(2u_i+v_i)(1-\eta)+u_i]^2},$$

$$\begin{aligned}\sigma_2(u_i, v_i, \eta) = &-\frac{\eta^3}{3} + \frac{\eta^2\{(3u_i+v_i) - \eta(2u_i+v_i) - 3u_i(1-\eta)\}}{3V_i(\eta)}\\ &+\frac{2\eta^3[\eta(2u_i+v_i) - \sqrt{H_i}]^3}{3[u_i+\eta(2u_i+v_i)]^3} + \frac{6u_i\eta^2(1-\eta)\{u_i+\eta(u_i+v_i) - \eta\sqrt{H_i}\}^2}{3V_i(\eta)\{u_i+\eta(2u_i+v_i)\}^2}\\ &-\frac{2\eta^2\{(3u_i+v_i) - \eta(2u_i+v_i)\}\{u_i+\eta(u_i+v_i) - \eta\sqrt{H_i}\}^3}{3V_i(\eta)\{u_i+\eta(2u_i+v_i)\}^3},\end{aligned}$$

$$H_i = u_i(u_i+v_i) + \eta v_i(2u_i+v_i),\ \sigma(u_i, v_i, \eta) = \begin{cases}\max\sigma_1(u_i, v_i, \eta), & 0 \le \eta \le \frac{u_i+v_i}{2u_i+v_i},\\ \max\sigma_2(u_i, v_i, \eta), & \frac{u_i+v_i}{2u_i+v_i} \le \eta \le 1,\end{cases}$$

$c_i^* := \max\{\sigma(u_i, v_i, \eta) : 0 \le \eta \le 1\}$, $i \in J$ *and* $c^* = \max\{c_i^* : i \in J\}$.

Proof. Let $\mathcal{F}^* = \{\phi : I \to \mathbb{R} |\ \phi \in \mathcal{C}^1(I),\ \phi(x_1) = f_1,\ \phi(x_N) = f_N,\ \phi^{(1)}(x_1) = d_1$ and $\phi^{(1)}(x_N) = d_N\}$. Then $(\mathcal{F}^*, \rho^*)$ is a complete metric space, where ρ^* is the metric induced by the $\mathcal{C}^1$ norm $\|\phi\| = \|\phi\|_\infty + \|(\phi)^{(1)}\|_\infty$ on $\mathcal{C}^1(I)$. The Read-Bajraktarević operator corresponding to the FIF Φ can be written as $T_\lambda^* : \mathcal{F}^* \to \mathcal{F}^*$ such that

$$(T_\lambda^*\phi)(x) = \lambda_i\phi(L_i^{-1}(x)) + \frac{p_i(L_i^{-1}(x), \lambda_i)}{q_i(L_i^{-1}(x))}, \quad x \in I_i,\ i \in J,$$

where $p_i(x, \lambda_i) \equiv P_i(\zeta)$ and $q_i(x) \equiv Q_i(\zeta)$ are as given in (3.3). It is known that Φ is the fixed point of the operator T_λ^* with $\lambda \neq \mathbf{0}$. Also, classical rational cubic spline C is the fixed point of T_λ^* with $\lambda = \mathbf{0} = (0, 0, \dots, 0) \in \mathbb{R}^{N-1}$. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{N-1})$ be a scale vector such that $|\lambda_i| < a_i$ for all $i \in J$ and with at least one $\lambda_i \neq 0$. For $\lambda \neq \mathbf{0}$, T_λ^* is a contraction map with uniform metric. Hence

$$\|T_\lambda^*\Phi - T_\lambda^*C\|_\infty \le |\lambda|_\infty\|\Phi - C\|_\infty. \tag{4.1}$$

Also for $x \in I_i$, we have

$$\begin{aligned}|T_\lambda^*C(x) - T_{\mathbf{0}}^*C(x)| &= \left|\lambda_i\ C \circ L_i^{-1}(x) + \frac{p_i(L_i^{-1}(x),\ \lambda_i)}{q_i(L_i^{-1}(x))} - \frac{p_i(L_i^{-1}(x),\ 0)}{q_i(L_i^{-1}(x))}\right|\\ &\le |\lambda_i|\|C\|_\infty + \frac{|p_i(L_i^{-1}(x),\ \lambda_i) - p_i(L_i^{-1}(x),\ 0)|}{q_i(L_i^{-1}(x))}.\end{aligned} \tag{4.2}$$

Using the mean-value theorem for functions of several variables, there exists $\beta = (\beta_1,\beta_2,\dots,\ \beta_{N-1})$ such that $|\beta_i| < |\lambda_i|$ and

$$p_i(L_i^{-1}(x),\ \lambda_i) - p_i(L_i^{-1}(x),\ 0) = \left(\frac{\partial}{\partial\lambda_i}(p_i(L_i^{-1}(x),\ \beta_i))\right)\lambda_i. \tag{4.3}$$

From (4.2) and (4.3), we can obtain

$$|T^*_\lambda C(x) - T^*_{\mathbf{0}} C(x)| \leq |\lambda_i| \left(\|C\|_\infty + \left| \frac{\partial}{\partial \lambda_i} \left(\frac{p_i(L_i^{-1}(x), \beta_i)}{q_i(L_i^{-1}(x))} \right) \right| \right). \tag{4.4}$$

To find the bound for the right hand side of (4.4), the classical rational cubic interpolant C can be written as

$$C(x) = w_1(u_i, v_i, \eta) f_i + w_2(u_i, v_i, \eta) f_{i+1} + w_3(u_i, v_i, \eta) d_i - w_4(u_i, v_i, \eta) d_{i+1}, \tag{4.5}$$

where

$$w_1(u_i, v_i, \eta) = \frac{u_i(1-\eta)^3 + (3u_i + v_i)\eta(1-\eta)^2}{V_i(\eta)}, \; w_3(u_i, v_i, \eta) = \frac{u_i h_i \eta (1-\eta)^2}{V_i(\eta)},$$

$$w_2(u_i, v_i, \eta) = \frac{(3u_i + v_i)\eta^2(1-\eta) + u_i \eta^3}{V_i(\eta)} \;\text{ and }\; w_4(u_i, v_i, \eta) = \frac{u_i h_i \eta^2 (1-\eta)}{V_i(\eta)}.$$

It can be observed that

$$w_1(u_i, v_i, \eta) + w_2(u_i, v_i, \eta) = 1.$$

Also, we get

$$\begin{aligned} w_3(u_i, v_i, \eta) + w_4(u_i, v_i, \eta) &= \frac{u_i h_i \eta(1-\eta)^2 + u_i h_i \eta^2 (1-\eta)}{u_i + v_i \eta (1-\eta)} \\ &\leq \frac{u_i h_i \eta(1-\eta)^2 + u_i h_i \eta^2 (1-\eta)}{u_i} \\ &= h_i \eta (1-\eta). \end{aligned}$$

From (4.5), we obtain

$$\begin{aligned} |C(x)| &\leq \max\{|f_i|, |f_{i+1}|\} + \frac{h_i}{4} \max\{|d_i|, |d_{i+1}|\} \\ &= M_i + \frac{h_i}{4} \overline{M_i}, \end{aligned}$$

where $M_i = \max\{|f_i|, |f_{i+1}|\}$ and $\overline{M_i} = \max\{|d_i|, |d_{i+1}|\}$.
Thus, we have

$$\|C\|_\infty \leq M + \frac{h}{4} \overline{M}. \tag{4.6}$$

It can be noticed that

$$\begin{aligned} \frac{\partial}{\partial \lambda_i} \left(\frac{p_i(L_i^{-1}(x), \beta_i)}{q_i(L_i^{-1}(x))} \right) = &- w_1^*(u_i, v_i, \eta) f_1 - w_2^*(u_i, v_i, \eta) f_N - w_3^*(u_i, v_i, \eta) d_1 \\ &+ w_4^*(u_i, v_i, \eta) d_N, \end{aligned}$$

where

$$w_1^*(u_i, v_i, \eta) = \frac{u_i(1-\eta)^3 + (3u_i + v_i)\eta(1-\eta)^2}{V_i(\eta)},$$
$$w_2^*(u_i, v_i, \eta) = \frac{(3u_i + v_i)\eta^2(1-\eta) + u_i\eta^3}{V_i(\eta)},$$
$$w_3^*(u_i, v_i, \eta) = \frac{u_i(x_N - x_1)\eta(1-\eta)^2}{V_i(\eta)}, \quad \text{and}$$
$$w_4^*(u_i, v_i, \eta) = \frac{u_i(x_N - x_1)\eta^2(1-\eta)}{V_i(\eta)}.$$

Thus, we have

$$\left|\frac{\partial}{\partial \lambda_i}\left(\frac{p_i(L_i^{-1}(x), \beta_i)}{q_i(L_i^{-1}(x))}\right)\right| \leq |w_1^*(u_i, v_i, \eta) f_1| + |w_2^*(u_i, v_i, \eta) f_N| + |w_3^*(u_i, v_i, \eta) d_1| + |w_4^*(u_i, v_i, \eta) d_N|.$$

By using a similar procedure for finding the bound for $\|C\|_\infty$, it is easy to see that

$$\left|\frac{\partial}{\partial \lambda_i}\left(\frac{p_i(L_i^{-1}(x), \beta_i)}{q_i(L_i^{-1}(x))}\right)\right| \leq M^* + \frac{|I|}{4} M_*. \tag{4.7}$$

From (4.4), (4.6) and (4.7), we obtain

$$|T_\lambda^* C(x) - T_{\mathbf{0}}^* C(x)| \leq |\lambda|_\infty([M + \frac{h}{4}\overline{M}] + [M^* + \frac{|I|}{4} M_*]), \text{ for } x \in I_i.$$

Since this inequality does not depend on the interval, we have

$$\|T_\lambda^* C - T_{\mathbf{0}}^* C\|_\infty \leq |\lambda|_\infty([M + \frac{h}{4}\overline{M}] + [M^* + \frac{|I|}{4} M_*]). \tag{4.8}$$

From (4.1) and (4.8), we get

$$\|\Phi - C\|_\infty = \|T_\lambda^* \Phi - T_{\mathbf{0}}^* C\|_\infty \leq \|T_\lambda^* \Phi - T_\lambda^* C\|_\infty + \|T_\lambda^* C - T_{\mathbf{0}}^* C\|_\infty,$$

which implies

$$\|\Phi - C\|_\infty \leq \frac{1}{(1 - |\lambda|_\infty)}\Big[|\lambda|_\infty\Big([M + \frac{h}{4}\overline{M}] + [M^* + \frac{|I|}{4} M_*]\Big)\Big]. \tag{4.9}$$

From [20], the error bound between original function S and the classical rational cubic spline C is

$$\|S - C\|_\infty \leq \frac{1}{2}\|S^{(3)}\|_\infty h^3 c^*. \tag{4.10}$$

Using (4.9) and (4.10) with the following inequality

$$\|S - \Phi\|_\infty \leq \|S - C\|_\infty + \|C - \Phi\|_\infty,$$

we get the required bound for $\|S - \Phi\|_\infty$. □

Remark 4.2. Since $|\lambda_i| < a_i = h_i/(x_N - x_1)$, $i \in J$, it can be seen that $\|S - \Phi\|_\infty = \mathrm{O}(h)$.

- If $|\lambda_i| < a_i^2$, then $\|S - \Phi\|_\infty = \mathrm{O}(h^2)$,
- If $|\lambda_i| < a_i^3$, then $\|S - \Phi\|_\infty = \mathrm{O}(h^3)$.

5. SHAPE PRESERVING ASPECTS OF FIF

Apart from interpolating the data, many times it is required that interpolant should satisfy the properties which inherited in the data. In this section, shape preserving aspects of the rational cubic FIF (3.3) are developed. There are different kinds of shape preserving aspects like positivity, monotonicity and convexity etc. Choosing random scaling factors and shape parameters, FIF (3.3) may not satisfy these properties. So it is required to find conditions on the scaling factors and the shape parameters to preserve these shape aspects.

5.1. POSITIVITY OF FIF

In the following theorem, the sufficient conditions on the scaling factors and the shape parameters are obtained to ensure the positivity of FIF.

Theorem 5.1. *Let $\{(x_i, f_i) : i = 1, 2, \ldots, N\}$ be a data such that $f_i > 0$, $i = 1, 2, \ldots, N$. Let d_i, $i = 1, 2, \ldots, N$ be the derivative value at the knot x_i. Then the following conditions on the scaling factors and the shape parameters are sufficient to the FIF (3.3) to satisfy the positivity conditions:*

$$0 \leq \lambda_i < \min\left\{a_i,\ \frac{f_i}{f_1},\ \frac{f_{i+1}}{f_N}\right\},$$

$$u_i > 0 \text{ and } v_i > \max\left\{0,\ \gamma_{1i},\ \gamma_{2i}\right\},$$

where

$$\gamma_{1i} = \frac{-u_i h_i d_i + \lambda_i u_i (x_N - x_1) d_1}{f_i - \lambda_i f_1},\ \gamma_{2i} = \frac{u_i h_i d_{i+1} - \lambda_i u_i (x_N - x_1) d_N}{f_{i+1} - \lambda_i f_N},\ i \in J.$$

Proof. The FIF Φ is positive if $\Phi(x) > 0$ for all $x \in [x_1, x_N]$. For each node x_j, $j = 1, 2, \ldots, N$ we obtain

$$\Phi(L_i(x_j)) = \lambda_i \Phi(x_j) + \frac{P_i(\zeta_j)}{Q_i(\zeta_j)},\ \ \zeta_j = \frac{x_j - x_1}{x_N - x_1},\ i \in J.$$

Assume that $\lambda_i \geq 0,\ i \in J$. Also $u_i > 0$ and $v_i > 0$ gives $Q_i(\zeta_j) > 0$. So $\Phi(L_i(x_j)) > 0,\ i \in J,\ j = 1, 2, \ldots, N$, if $P_i(\zeta_j) > 0$. Now, we have

$$P_i(\zeta_j) = A_{1,i}(1-\zeta_j)^3 + A_{2,i}\zeta_j(1-\zeta_j)^2 + A_{3,i}{\zeta_j}^2(1-\zeta_j) + A_{4,i}{\zeta_j}^3.$$

It can be seen that $P_i(\zeta_j) > 0$ if $A_{1,i} > 0,\ A_{2,i} > 0,\ A_{3,i} > 0$ and $A_{4,i} > 0$. We get

$$A_{1,i} > 0 \text{ if } \lambda_i < \frac{f_i}{f_1} \text{ and } A_{4,i} > 0 \text{ if } \lambda_i < \frac{f_{i+1}}{f_N}.$$

Let $0 \leq \lambda_i < \left\{\frac{f_i}{f_1},\ \frac{f_{i+1}}{f_N}\right\}$. Then we have

$$A_{2,i} > 0 \quad \text{if } v_i > \frac{-u_i h_i d_i + \lambda_i u_i (x_N - x_1) d_1}{f_i - \lambda_i f_1}.$$

Also, we get

$$A_{3,i} > 0 \quad \text{if } v_i > \frac{u_i h_i d_{i+1} - \lambda_i u_i (x_N - x_1) d_N}{f_{i+1} - \lambda_i f_N}.$$

From the above results, it is clear that $\Phi(L_i(x_j)) > 0$ for all $i \in J$, $j = 1, 2, \ldots, N$ if the scaling factors and the shape parameters satisfy the sufficient conditions given in the statement of Theorem 5.1. Since the FIF has recursive nature, therefore, $\Phi(L_i(x_j)) > 0$, for all $i \in J,\ j = 1, 2, \ldots, N$ which in turn $\Phi(x) > 0$ for all $x \in [x_1,\ x_N]$. $\square$

Remark 5.2. If $\lambda_i = 0,\ i \in J$, FIF (3.3) reduces to the classical rational cubic spline (3.6). Then

$$u_i > 0 \text{ and } v_i > \max\left\{0, \frac{-u_i h_i d_i}{f_i},\ \frac{u_i h_i d_{i+1}}{f_{i+1}}\right\}$$

become the sufficient conditions for the classical rational spline to be positive.

5.2. MONOTONICITY OF FIF

Let $\{(x_i, f_i) : i = 1, 2, \ldots, N\}$ be a monotonic data. Let d_i be the derivative value of the unknown function at the knot x_i. Without loss of generality assume that the data is monotonically increasing i.e., $f_1 \leq f_2 \leq \cdots \leq f_N$. Then $\Delta_i = (f_{i+1} - f_i)/(x_{i+1} - x_i) \geq 0,\ i \in J$. From calculus, Φ is monotonically increasing in $[x_1, x_N]$ if $\Phi^{(1)}(x) \geq 0$, for all $x \in [x_1, x_N]$. We have

$$\Phi^{(1)}(L_i(x)) = \frac{\lambda_i \Phi^{(1)}(x)}{a_i} + \frac{\Psi_i(\zeta)}{(Q_i(\zeta))^2},$$

where $\Psi_i(\zeta) = \sum_{k=1}^{5} B_{k,i}\zeta^{k-1}(1-\zeta)^{5-k}$,

$$B_{1,i} = u_i^2 d_i^*,$$

$$B_{2,i} = 2u_i\{(3u_i+v_i)\Delta_i^* - u_i d_{i+1}^*\},$$

$$B_{3,i} = 3u_i^2\Delta_i^* + (3u_i+v_i)\{(3u_i+v_i)\Delta_i^* - u_i d_{i+1}^* - u_i d_i^*\},$$

$$B_{4,i} = 2u_i\{(3u_i+v_i)\Delta_i^* - u_i d_i^*\},$$

$$B_{5,i} = u_i^2 d_{i+1}^*,$$

$$d_i^* = d_i - \frac{\lambda_i d_1}{a_i},\ d_{i+1}^* = d_{i+1} - \frac{\lambda_i d_N}{a_i} \text{ and } \Delta_i^* = \Delta_i - \lambda_i \frac{f_N - f_1}{h_i}.$$

Theorem 5.3. *Let $\{(x_i, f_i, d_i) : i = 1, 2, \ldots, N\}$ be a monotonically increasing data. Let the derivative values satisfy the necessary condition for monotonicity, i.e., $sgn(d_i) = sgn(d_{i+1}) = sgn(\Delta_i)$. Then the following conditions on the scaling factors and the shape parameters are sufficient to the FIF Φ defined in (3.3) to satisfy monotonicity:*

$$0 \le \lambda_i < \left\{a_i,\ \frac{a_i d_i}{d_1},\ \frac{a_i d_{i+1}}{d_N},\ \frac{f_{i+1} - f_i}{f_N - f_1}\right\},$$

$$u_i > 0 \text{ and } v_i > \max\left\{0,\ \frac{u_i(d_i^* + d_{i+1}^*)}{\Delta_i^*}\right\},\ i \in J.$$

Proof. $\Phi(x)$ is monotonically increasing, if $\Phi^{(1)}(x) \ge 0,\ x \in [x_1, x_N]$. For each node $x_j,\ j = 1,2,\ldots,N$, we have

$$\Phi^{(1)}(L_i(x_j)) = \frac{\lambda_i \Phi^{(1)}(x_j)}{a_i} + \frac{\Psi_i(\zeta_j)}{(Q_i(\zeta_j))^2}, \quad \zeta_j = \frac{x_j - x_1}{x_N - x_1},\ i \in J.$$

Assume $\lambda_i \ge 0,\ i \in J$. It can be seen that $(Q_i(\zeta_j))^2 > 0$. Therefore, $\Phi^{(1)}(L_i(x_j)) \ge 0$, if $\Psi_i(\zeta_j) \ge 0$. We have

$$\Psi_i(\zeta_j) = B_{1,i}(1-\zeta_j)^4 + B_{2,i}\zeta_j(1-\zeta_j)^3 + B_{3,i}{\zeta_j}^2(1-\zeta_j)^2 + B_{4,i}{\zeta_j}^3(1-\zeta_j) + B_{5,i}\zeta_j^4.$$

It can be seen that $\Psi_i(\zeta_j) \ge 0$, if $B_{1,i} \ge 0,\ B_{2,i} \ge 0,\ B_{3,i} \ge 0,\ B_{4,i} \ge 0$ and $B_{5,i} \ge 0$. We have

$$B_{1,i} \ge 0 \text{ if } \lambda_i \le \frac{a_i d_i}{d_1},\ B_{5,i} \ge 0 \text{ if } \lambda_i \le \frac{a_i d_{i+1}}{d_N}.$$

Let $0 \le \lambda_i < \left\{\frac{a_i d_i}{d_1},\ \frac{a_i d_{i+1}}{d_N},\ \frac{f_{i+1} - f_i}{f_N - f_1}\right\}$. We get

$$B_{2,i} \ge 0 \text{ if } v_i \ge \frac{u_i d_{i+1}^*}{\Delta_i^*},\ B_{3,i} \ge 0 \text{ if } v_i \ge \frac{u_i(d_i^* + d_{i+1}^*)}{\Delta_i^*},\ B_{5,i} \ge 0 \text{ if } v_i \ge \frac{u_i d_i^*}{\Delta_i^*}.$$

So according to the conditions prescribed in Theorem 5.3, it is clear that $\Phi^{(1)}(L_i(x_j)) \ge 0$ for all $j \in J,\ i = 1, 2, \ldots, N$. Since $\Phi^{(1)}$ is also a fractal function and it has recursive nature, the condition $\Phi^{(1)}(L_i(x_j)) \ge 0, i \in J,\ j = 1, 2, \ldots, N$ gives $\Phi^{(1)}(x) \ge 0$ for all $x \in [x_1, x_N]$. $\square$

Remark 5.4. If $\lambda_i = 0,\ i \in J$ then the sufficient conditions for classical rational cubic spline (3.6) which preserves the monotonicity are

$$u_i > 0 \text{ and } v_i > \max\Big\{0,\ \frac{u_i(d_i + d_{i+1})}{\Delta_i}\Big\},\ i \in J.$$

5.3. CONVEXITY OF FIF

A data $\{(x_i, f_i) : i = 1, 2, \ldots, N\}$ is said to be convex if $\Delta_1 \leq \Delta_2 \leq \cdots \leq \Delta_{i-1} \leq \Delta_i \leq \cdots \leq \Delta_{N-1}$. Assume that the data $\{(x_i, f_i) : i = 1, 2, ..., N\}$ is strictly convex. To avoid the possibility of straight line segments, assume that $d_1 < \Delta_1 < ... < d_i < \Delta_i < d_{i+1} < ... < \Delta_{N-1} < d_N$. In this section, sufficient conditions on the scaling factors and the shape parameters will be determined to ensure the convexity of the FIF (3.3).

Theorem 5.5. *Suppose $\{(x_i, f_i) : i = 1, 2, ..., N\}$ is a strictly convex data. Let d_i be the derivative value at the knot x_i. Let the derivative values satisfy $d_1 < \Delta_1 < ... < d_i < \Delta_i < d_{i+1} < ... < \Delta_{N-1} < d_N$. Then sufficient conditions on the scaling factors and the shape parameters to ensure convexity of Φ in (3.3) are*

$$0 \leq \lambda_i < \min\Big\{a_i^2, \frac{h_i(d_{i+1} - \Delta_i)}{d_N(x_N - x_1) - (f_N - f_1)}, \frac{h_i(\Delta_i - d_i)}{(f_N - f_1) - d_1(x_N - x_1)}, \frac{a_i(d_{i+1} - d_i)}{(d_N - d_1)}\Big\},$$

$$u_i > 0 \textit{ and } v_i > \max\Big\{\frac{u_i(d^*_{i+1} - \Delta^*_i)}{(\Delta^*_i - d^*_i)},\ \frac{u_i(\Delta^*_i - d^*_i)}{(d^*_{i+1} - \Delta^*_i)}\Big\}.$$

Proof. Since Φ belongs to $\mathcal{C}^1$, Φ is convex, if $\Phi^{(2)}(x^+)$ or $\Phi^{(2)}(x^-)$ exists and non-negative for all $x \in (x_1, x_N)$[4, 17]. We have

$$\Phi^{(2)}(L_i(x)) = \frac{\lambda_i \Phi^{(2)}(x)}{a_i^2} + \frac{\Psi^*_i(\zeta)}{h_i(Q_i(\zeta))^3},$$

where $\Psi^*_i(\zeta) = \sum_{k=1}^{6} C_{k,i}\zeta^{k-1}(1-\zeta)^{6-k}$,

$$C_{1,i} = 2u_i^2\{(2u_i + v_i)(\Delta^*_i - d^*_i) - u_i(d^*_{i+1} - \Delta^*_i)\},$$

$$C_{2,i} = 2u_i^2\{7u_i(\Delta^*_i - d^*_i) + 2v_i(\Delta^*_i - d^*_i) - 2u_i(d^*_{i+1} - \Delta^*_i)\},$$

$$C_{3,i} = 2u_i\{(6u_i^2 + u_iv_i)(\Delta^*_i - d^*_i) + 2u_i^2(d^*_{i+1} - d^*_i)\},$$

$$C_{4,i} = 2u_i\{(6u_i^2 + u_iv_i)(d^*_{i+1} - \Delta^*_i) + 2u_i^2(d^*_{i+1} - d^*_i)\},$$

$$C_{5,i} = 2u_i^2\{7u_i(d^*_{i+1} - \Delta^*_i) + 2v_i(d^*_{i+1} - \Delta^*_i) - 2u_i(\Delta^*_i - d^*_i)\},$$

$$C_{6,i} = 2u_i^2\{(2u_i + v_i)(d^*_{i+1} - \Delta^*_i) - u_i(\Delta^*_i - d^*_i)\},$$

$d_i^* = d_i - \frac{\lambda_i d_1}{a_i},\ d_{i+1}^* = d_{i+1} - \frac{\lambda_i d_N}{a_i},\ \Delta_i^* = \Delta_i - \lambda_i \frac{f_N - f_1}{h_i}$.
Now, we get

$$\Phi^{(2)}(x_1^+) = \frac{C_{1,1}}{h_1 u_1^3}\Big[1 - \frac{\lambda_1}{a_1^2}\Big]^{-1}, \tag{5.1}$$

$$\Phi^{(2)}(x_N^-) = \frac{C_{6,N-1}}{h_{N-1} u_{N-1}^3}\Big[1 - \frac{\lambda_{N-1}}{a_{N-1}^2}\Big]^{-1}, \tag{5.2}$$

$$\Phi^{(2)}(x_n^+) = \frac{\lambda_n}{a_n^2}\Phi^{(2)}(x_1^+) + \frac{C_{1,n}}{u_n^3 h_n}, \quad n = 2, 3, \ldots, N-1. \tag{5.3}$$

Let $0 \le \lambda_i < a_i^2,\ i \in J$. From (5.1), (5.2) and (5.3), it is evident that if $C_{1,i} \ge 0,\ i \in J$ and $C_{6,N-1} \ge 0$, then the right-handed second derivatives at the knots $x_i,\ i \in J$ and the left-handed second derivative at x_N are nonnegative. For a knot point $x_n,\ n \in J$ we get

$$\Phi^{(2)}(L_i(x_n^+)) = \frac{\lambda_i \Phi^{(2)}(x_n^+)}{a_i^2} + R_i(x_n^+),\ i \in J,$$

where $R_i(x) = R_i(x_1 + \zeta(x_N - x_1)) = \Psi_i^*(\zeta)/(h_i(Q_i(\zeta))^3)$. Assuming that $C_{1,i} \ge 0,\ i \in J$, we get $\Phi^{(2)}(L_i(x_n^+)) \ge 0$ if $R_i(x_n^+) \ge 0$. Note that $R_i(x_n^+) \ge 0$, if the coefficients $C_{j,i} \ge 0$, for $j = 1, 2, \ldots, 6$. Using the Three chords lemma for the convex functions [4], the convex data should satisfy $d_1 < (f_N - f_1)/(x_N - x_1) < d_N$. One can see that

$$\begin{aligned}
(\Delta_i^* - d_i^*) > 0, &\quad \text{if } \lambda_i < \frac{h_i(\Delta_i - d_i)}{(f_N - f_1) - d_1(x_N - x_1)},\\
(d_{i+1}^* - \Delta_i^*) > 0, &\quad \text{if } \lambda_i < \frac{h_i(d_{i+1} - \Delta_i)}{d_N(x_N - x_1) - (f_N - f_1)},\\
(d_{i+1}^* - d_i^*) > 0, &\quad \text{if } \lambda_i < \frac{a_i(d_{i+1} - d_i)}{(d_N - d_1)}.
\end{aligned}$$

Let

$$0 \le \lambda_i < \min\Big\{\frac{h_i(d_{i+1} - \Delta_i)}{d_N(x_N - x_1) - (f_N - f_1)}, \frac{h_i(\Delta_i - d_i)}{(f_N - f_1) - d_1(x_N - x_1)}, \frac{a_i(d_{i+1} - d_i)}{(d_N - d_1)}\Big\}. \tag{5.4}$$

It can be seen that

$$C_{1,i} \ge 0 \quad \Leftrightarrow \quad (2u_i + v_i)(\Delta_i^* - d_i^*) - u_i(d_{i+1}^* - \Delta_i^*) \ge 0.$$

$$(2u_i + v_i)(\Delta_i^* - d_i^*) - u_i(d_{i+1}^* - \Delta_i^*) \ge 0, \quad \text{if } v_i \ge \frac{u_i(d_{i+1}^* - \Delta_i^*)}{(\Delta_i^* - d_i^*)}.$$

$$C_{2,i} \ge 0 \quad \Leftrightarrow \quad 7u_i(\Delta_i^* - d_i^*) + 2v_i(\Delta_i^* - d_i^*) - 2u_i(d_{i+1}^* - \Delta_i^*) \ge 0.$$

$$7u_i(\Delta_i^* - d_i^*) + 2v_i(\Delta_i^* - d_i^*) - 2u_i(d_{i+1}^* - \Delta_i^*) \ge 0, \quad \text{if } v_i \ge \frac{u_i(d_{i+1}^* - \Delta_i^*)}{(\Delta_i^* - d_i^*)}.$$

$$C_{3,i} \ge 0 \quad \Leftrightarrow \quad (6u_i^2 + u_i v_i)(\Delta_i^* - d_i^*) + 2u_i^2(d_{i+1}^* - d_i^*) \ge 0.$$

The assumption on the scaling factor given in (5.4) ensures that $C_{3,i} \geq 0$.

$$C_{4,i} \geq 0 \quad \Leftrightarrow \quad (6u_i^2 + u_i v_i)(d_{i+1}^* - \Delta_i^*) + 2u_i^2(d_{i+1}^* - d_i^*) \geq 0.$$

The assumption on the scaling factor given in (5.4) shows that $C_{4,i} \geq 0$. Similarly, one can observe that

$$C_{5,i} \geq 0, \quad \text{if } v_i \geq \frac{u_i(\Delta_i^* - d_i^*)}{(d_{i+1}^* - \Delta_i^*)},$$
$$C_{6,i} \geq 0, \quad \text{if } v_i \geq \frac{u_i(\Delta_i^* - d_i^*)}{(d_{i+1}^* - \Delta_i^*)}.$$

Thus the conditions on the scaling factors and the shape parameters given in Theorem 5.5 ensure $C_{j,i} \geq 0$, $i \in J$, $j = 1, 2, \dots, 6$ and hence the nonnegativity of $\Phi^{(2)}(L_i(x_n^+))$ for $i, n \in J$, $\Phi^{(2)}(x_N^-)$. The nonnegativity of $\Phi^{(2)}(L_i(x_n^+))$ for $i, n \in J$, and $\Phi^{(2)}(x_N^-)$ ensures $\Phi^{(2)}(x^+)$ or $\Phi^{(2)}(x^-)$ for $x \in (x_1, x_N)$. $\square$

Remark 5.6. If $\lambda_i = 0$, then the conditions

$$u_i > 0 \text{ and } v_i > \max\left\{\frac{u_i(d_{i+1} - \Delta_i)}{(\Delta_i - d_i)}, \frac{u_i(\Delta_i - d_i)}{(d_{i+1} - \Delta_i)}\right\},$$

are the sufficient conditions for the classical rational cubic spline (3.6) to be convex.

Remark 5.7. If $\Delta_i - d_i = 0$ or $d_{i+1} - \Delta_i = 0$, then take $\lambda_i = 0$, $d_i = d_{i+1} = \Delta_i$. In this case, Φ becomes a straight line $\Phi(L_i(x)) = f_i(1 - \zeta) + f_{i+1}\zeta$ in the interval $[x_i, x_{i+1}]$.

5.4. CONSTRAINED FIF

Let $\{(x_i, f_i) : i = 1, 2, \dots, N\}$ be the data such that it lies above the straight line $y = mx + c$, i.e., $f_i > y_i$ where $y_i = mx_i + c$. Then, in general, the FIF Φ may not lie above the line $y = mx + c$ in $[x_1, x_N]$. In this subsection, sufficient conditions on the scaling factors and the shape parameters are derived such that Φ would lie above the straight line $y = mx + c$. The straight line $y = mx + c$ can be written as $y_1(1 - \zeta) + y_N\zeta$, $\zeta = (x - x_1)/(x_1 - x_N)$, $x \in [x_1, x_N]$. The FIF Φ lies above straight line $y_1(1 - \zeta) + y_N\zeta$ if $\Phi(x) > y_1(1 - \zeta) + y_N\zeta$, for all $x \in [x_1, x_N]$. Since the graph of Φ is the attractor of the IFS (3.5), it is evident that $\Phi(x) > y_1(1 - \zeta) + y_N\zeta$, for all $x \in [x_1, x_N]$ if $F_i(x, f) > y_i(1 - \zeta) + y_{i+1}\zeta$ for all (x, f) such that $x \in [x_1, x_N]$, $f > y_1(1 - \zeta) + y_N\zeta$, $i = 1, 2, \dots, N - 1$.

Theorem 5.8. *Let $\{(x_i, f_i) : i = 1, 2, \dots, N\}$ be the data such that it lies above the straight line $y = mx + c$, i.e., $f_i > y_i$ where $y_i = mx_i + c$. Then the FIF Φ defined in*

(3.3) lies above the straight line $y = mx + c$ in $[x_1, x_N]$, if the scaling factors and the shape parameters satisfy the following conditions:

$$0 \le \lambda_i < \min\left\{a_i,\ \frac{f_i - y_i}{f_1 - y_1},\ \frac{f_{i+1} - y_{i+1}}{f_N - y_N}\right\},$$

$$u_i > 0 \text{ and } v_i > \max\left\{0, \delta_{1,i},\ \delta_{2,i}\right\},$$

where

$$\delta_{1,i} = \frac{-u_i[(f_i - y_{i+1}) - \lambda_i(f_1 - y_N)] - u_i[h_i d_i - \lambda_i(x_N - x_1)d_1]}{(f_i - y_i) - \lambda_i(f_1 - y_1)},$$

$$\delta_{2,i} = \frac{-u_i[(f_{i+1} - y_i) - \lambda_i(f_N - y_1)] - u_i[-h_i d_{i+1} + \lambda_i(x_N - x_1)d_N]}{(f_{i+1} - y_{i+1}) - \lambda_i(f_N - y_N)},$$

$i = 1, 2, \dots, N-1$.

Proof. For each fixed $i = 1, 2, \dots, N-1$, let $0 \le \lambda_i < a_i$ and (x, f) such that $x \in [x_1, x_N]$ and $f > y_1(1-\zeta) + y_N\zeta$. It is evident that $f\lambda_i > [y_1(1-\zeta) + y_N\zeta]\lambda_i$. We have

$$\lambda_i f + \frac{P_i(\zeta)}{Q_i(\zeta)} > [y_1(1-\zeta) + y_N\zeta]\lambda_i + \frac{P_i(\zeta)}{Q_i(\zeta)}.$$

To prove $F_i(x, f) > y_i(1-\zeta) + y_{i+1}\zeta$, it is enough to prove that

$$[y_1(1-\zeta) + y_N\zeta]\lambda_i + \frac{P_i(\zeta)}{Q_i(\zeta)} > y_i(1-\zeta) + y_{i+1}\zeta, \tag{5.5}$$

Now (5.5) can be written as

$$\frac{Q_i(\zeta)[(\lambda_i y_1 - y_i)(1-\zeta) + (\lambda_i y_N - y_{i+1})\zeta] + P_i(\zeta)}{Q_i(\zeta)} > 0. \tag{5.6}$$

Since $Q_i(\zeta) > 0$, to prove (5.6), it is sufficient to show that $Q_i(\zeta)[(\lambda_i y_1 - y_i)(1-\zeta) + (\lambda_i y_N - y_{i+1})\zeta] + P_i(\zeta) > 0$. The numerator $Q_i(\zeta)[(\lambda_i y_1 - y_i)(1-\zeta) + (\lambda_i y_N - y_{i+1})\zeta] + P_i(\zeta)$ can be written as $Q_i(\zeta)[(\lambda_i y_1 - y_i)(1-\zeta) + (\lambda_i y_N - y_{i+1})\zeta] + P_i(\zeta) = D_{1,i}(1-\zeta)^3 + D_{2,i}\zeta(1-\zeta)^2 + D_{3,i}\zeta^2(1-\zeta) + D_{4,i}\zeta^3$, where

$$\begin{aligned}
D_{1,i} =& u_i[(f_i - y_i) - \lambda_i(f_1 - y_1)],\ D_{4,i} = u_i[(f_{i+1} - y_{i+1}) - \lambda_i(f_N - y_N)],\\
D_{2,i} =& (2u_i + v_i)[(f_i - y_i) - \lambda_i(f_1 - y_1)] + u_i[(f_i - y_{i+1}) - \lambda_i(f_1 - y_N)]\\
&+ u_i[h_i d_i - \lambda_i(x_N - x_1)d_1],\\
D_{3,i} =& (2u_i + v_i)[(f_{i+1} - y_{i+1}) - \lambda_i(f_N - y_N)] + u_i[(f_{i+1} - y_i) - \lambda_i(f_N - y_1)]\\
&+ u_i[-h_i d_{i+1} + \lambda_i(x_N - x_1)d_N].
\end{aligned}$$

It can be seen that $Q_i(\zeta)[(\lambda_i y_1 - y_i)(1-\zeta) + (\lambda_i y_N - y_{i+1})\zeta] + P_i(\zeta) > 0$, if all the coefficients $D_{j,i} > 0$, $j = 1, 2, \ldots, 4$. We have

$$D_{1,i} > 0, \quad \text{if } \lambda_i < \frac{f_i - y_i}{f_1 - y_1}, \qquad D_{4,i} > 0, \text{ if } \lambda_i < \frac{f_{i+1} - y_{i+1}}{f_N - y_N}.$$

Let $0 \le \lambda_i < \min\left\{\frac{f_i - y_i}{f_1 - y_1},\ \frac{f_{i+1} - y_{i+1}}{f_N - y_N}\right\}$. The coefficient $D_{2,i} > 0$, if

$$v_i > \frac{-u_i[(f_i - y_{i+1}) - \lambda_i(f_1 - y_N)] - u_i[h_i d_i - \lambda_i(x_N - x_1)d_1]}{(f_i - y_i) - \lambda_i(f_1 - y_1)}.$$

The coefficient $D_{3,i} > 0$, if

$$v_i > \frac{-u_i[(f_{i+1} - y_i) - \lambda_i(f_N - y_1)] - u_i[-h_i d_{i+1} + \lambda_i(x_N - x_1)d_N]}{(f_{i+1} - y_{i+1}) - \lambda_i(f_N - y_N)}.$$

Thus the sufficient conditions on the scaling factors and the shape parameters to satisfy (5.5) are

$$0 \le \lambda_i < \min\left\{a_i,\ \frac{f_i - y_i}{f_1 - y_1},\ \frac{f_{i+1} - y_{i+1}}{f_N - y_N}\right\},$$

$$u_i > 0 \text{ and } v_i > \max\left\{\delta_{1,i},\ \delta_{2,i}\right\},$$

$i = 1, 2, \ldots, N-1$. □

6. NUMERICAL EXAMPLES

In this section, the effectiveness of the FIF (3.3) towards the visualization of the shaped data is illustrated through numerical examples.

Example 6.1. Consider the positive data set $\{\ (1,14), (2,8), (3,2), (8,0.8), (10,0.5), (11,0.25), (12,0.4), (14,0.37)\ \}$as used in [20]. Using the arithmetic mean method, the derivatives are approximated and are given by $d_1 = -6.0000$, $d_2 = -6.0000$, $d_3 = -5.0400$, $d_4 = -0.1757$, $d_5 = -0.2167$, $d_6 = -0.0500$, $d_7 = 0.0950$ and $d_8 = -0.1250$.

Figure 2(a) represents a nonpositive FIF which is drawn based on an arbitrary scaling factors and shape parameters. According to restrictions given in Theorem 5.1, the scaling factors and shape parameters are restricted. Figure 2(b) is a positive FIF which is drawn based on these restrictions.

To demonstrate the effect of the scaling factors on the FIF given in Figure 2(b), by perturbing the scaling factors λ_3 and λ_4, the FIF in Figure 2(c) is constructed. There are significant changes in Figure 2(c) in the intervals $[x_3, x_4]$ and $[x_4, x_5]$ in which the curve becomes smooth and moves upward. Whereas the changes in the other intervals are not significantly visible.

To demonstrate the effect of shape parameters on the FIF given in Figure 2(b), the FIF in Figure 2(d) is constructed by changing the shape parameters v_3 and v_4. There is a little change in the intervals $[x_3, x_4]$ and $[x_4, x_5]$, but there is no significant changes in the other intervals.

Figure 2(e) is constructed by perturbing the scaling factors λ_5, λ_6 and λ_7 with respect to Figure 2(b). Observe that the FIF becomes smooth in the corresponding intervals. Finally, the classical rational cubic spline is constructed by taking all scaling factors as zero and it is shown in Figure 2(f). The values of scaling factors and shape parameters used in drawing various FIFs in Figure 2 are given in Table 2.

Example 6.2. Next consider the monotonic data set $\{ (0, 0.5), (2.5, 1.61), (3, 7.3891), (6, 9.8696), (11, 22.18), (15, 27.3), (20, 35.2) \}$ [8]. The derivatives are computed using arithmetic mean method and are given by $d_1 = 0$, $d_2 = 9.7058$, $d_3 = 10.0251$, $d_4 = 1.4401$, $d_5 = 1.8054$, $d_6 = 1.4133$ and $d_7 = 1.7467$. According to restrictions given in Theorem 5.3, the scaling factors and the shape parameters are restricted.

Figure 3(a) represents a monotonic FIF which passes through a monotonic data. To demonstrate the effect of the scaling factors on the FIF given in Figure 3(a), by perturbing the scaling factors λ_1 and λ_3, the FIF in Figure 3(b) is constructed. Visible changes occurred in the intervals $[x_1, x_2]$ and $[x_3, x_4]$, whereas changes in the other intervals are not visible.

Similarly by perturbing the scaling factors λ_4, λ_5 and λ_6 with respect to Figure 3(a), Figure 3(c) is constructed. Curve between the corresponding intervals moving towards a straight line. To demonstrate the effect of the shape parameters on the FIF in Figure 3(a), the FIF in Figure 3(d) is constructed by changing the shape parameters v_3 and v_4. In similar fashion, we perturb the shape parameters v_5 and v_6 with respect to Figure 3(a), to construct Figure 3(e). Finally by taking all the scaling factors are zero, classical rational cubic spline is constructed and is shown in Figure 3(f). The values of the scaling factors and the shape parameters used in drawing various FIFs in Figure 3 are given in Table 3.

Example 6.3. Next consider the convex data $\{ (1, 10), (2, 2.5), (4, 0.625), (5, 0.4), (10, 0.1) \}$ as given in [20]. Using arithmetic mean method, the derivatives are approximated and are given by $d_1 = -9.6875$, $d_2 = -5.3125$, $d_3 = -0.4625$, $d_4 = -0.1975$ and $d_5 = 0$.

Figure 4(a) represents a nonconvex FIF which is drawn based on an arbitrary scaling factors and shape parameters. According to the restrictions given in Theorem 5.5, the scaling factors and the shape parameters are restricted. Figure 4(b) is a convex FIF which is drawn based on these restrictions.

To demonstrate the effect of the scaling factors on the FIF in Figure 4(b), by perturbing the scaling factor λ_2, the FIF in Figure 4(c) is drawn. From Figure 4(c) it is evident that significant change occurred in the interval $[x_2, x_3]$ i.e., the curve becomes more smooth in this interval and changes in other intervals are negligible.

To demonstrate the effect of the shape parameters on the FIF in Figure 4(b), by perturbing shape parameter v_2, the FIF in Figure 4(d) is constructed. In the interval $[x_2, x_3]$, curve moves upward and whereas in other intervals the changes are not visible. By taking all the scaling factors are zero, Figure 4(e) is drawn which is classical rational cubic spline preserving convexity. By taking arbitrary scaling factors and shape parameters Figure 4(f) is drawn. The values of the scaling factors and the shape parameters used in drawing various FIFs in Figure 4 are given in Table 4.

Example 6.4. Next, by considering the interpolation data $\{\ (1, 2.5),\ (1.25, 1.5),\ (2.8, 2),\ (3, 2.5),\ (3.2, 3.5), (4.2, 4.5),\ (4.5, 5, 5)\ \}$ which lies above the line $y = 0.5x + 0.28$, the FIF is constructed. By taking arbitrary scaling factors and shape parameters Figure 5(a) is constructed. From Figure 5(a), it is clear that FIF does not lying above the line $y = 0.5x + 0.28$. Therefore, the scaling factors and the shape parameters are taken according to Theorem 5.8 and using this, Figure 5(b) is drawn. In Figure 5(b), it is observed that the FIF lies above the line $y = 0.5x+0.28$. After perturbing scaling factor λ_5 with respect to Figure 5(b), Figure 5(c) is constructed. In a similar fashion, after perturbing the shape parameter v_5 with respect to Figure 5(b), Figure 5(d) is drawn. Classical constrained FIF is constructed by taking all the scaling factors are zero which is shown in Figure 5(e). By taking random scaling factors and shape parameters, Figure 5(f) is constructed. The parameters used to construct Figure 5 are given in Table 5.

7. CONCLUSION

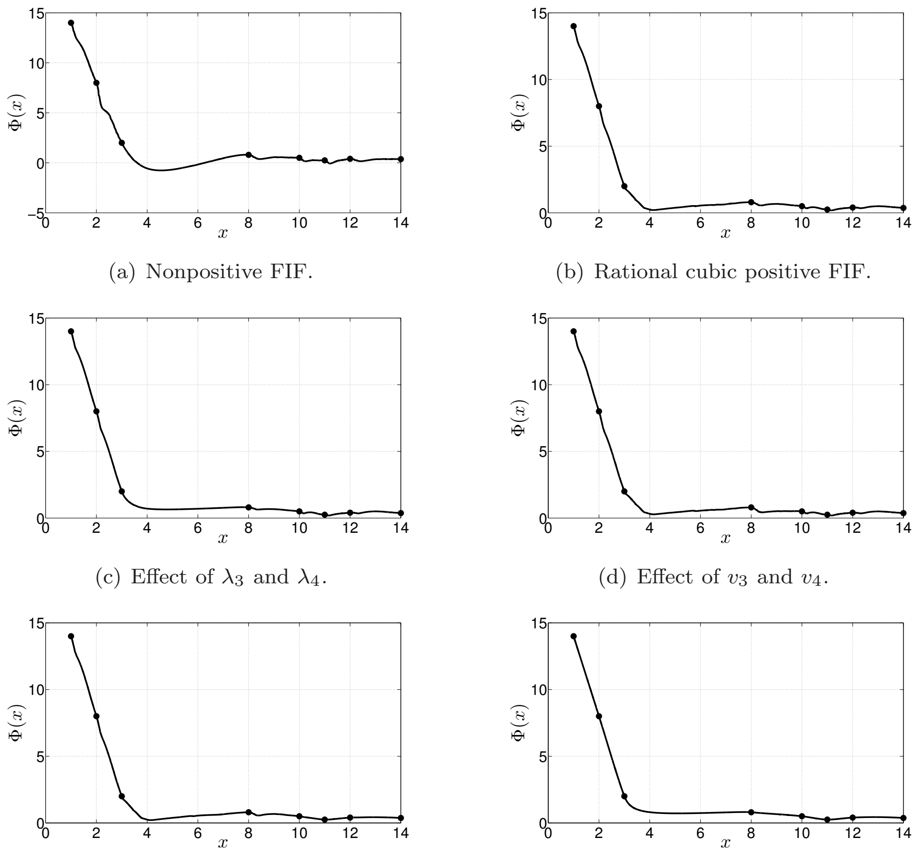
To interpolate the data, a new rational cubic fractal interpolation function is thus constructed with the help of rational spline. The conditions derived on the scaling factors and half of the shape parameters to preserve the shapes of the data makes it computationally economical. By perturbing the scaling factors and the shape parameters, shape of the curve can be modified according to desire. Therefore, the constructed FIF has more influence on shape preserving problem. By imposing suitable conditions on the scaling factors, the FIF has $O(h^3)$ accuracy if the original function belongs to the class $\mathcal{C}^3$.

Table 2: Parameters for positive FIFs with $u_i = 1.5$ for $i = 1, 2, \dots, 7$.

parameter	Figure 2(a)	Figure 2(b)	Figure 2(c)	Figure 2(d)	Figure 2(e)	Figure 2(f)
λ_1	0.1000	0.0754	0.0754	0.0754	0.0754	0
λ_2	0.2300	0.0754	0.0754	0.0754	0.0754	0
λ_3	0.0300	0.1324	0.0132	0.1324	0.1324	0
λ_4	0.0600	0.0471	0.0271	0.0471	0.0471	0
λ_5	0.0420	0.0347	0.0347	0.0347	0.0034	0
λ_6	0.0500	0.0169	0.0169	0.0169	0.0016	0
λ_7	0.0450	0.0266	0.0266	0.0266	0.0026	0
v_1	0.5000	0.5000	0.5000	0.5000	0.5000	0.8000
v_2	3.5000	1.5000	1.5000	1.5000	1.5000	1.5000
v_3	4.0000	153.0000	20.9733	350.0000	153.0000	20.9000
v_4	5.6000	2.5000	2.5000	10.5000	2.5000	1.5000
v_5	8.5000	1.0000	1.0000	1.0000	4.0000	1.0000
v_6	7.9000	3.0000	3.0000	3.0000	7.3000	3.0000
v_7	5.0000	0.5000	0.5000	0.5000	2.0000	0.5000

Table 3: Parameters for monotonic FIFs with $u_i = 1.5$ for $i = 1, 2, \dots, 6$.

parameter	Figure 3(a)	Figure 3(b)	Figure 3(c)	Figure 3(d)	Figure 3(e)	Figure 3(f)
λ_1	0.0250	0.0015	0.0250	0.0250	0.0250	0
λ_2	0.0240	0.0240	0.0240	0.0240	0.0240	0
λ_3	0.0610	0.0210	0.0610	0.0610	0.0610	0
λ_4	0.2000	0.2000	0.1000	0.2000	0.2000	0
λ_5	0.1360	0.1360	0.0940	0.1360	0.1360	0
λ_6	0.2160	0.2160	0.1160	0.2160	0.2160	0
v_1	145.7800	35.1300	145.7800	145.7800	145.7800	33.7900
v_2	3.7300	3.7300	3.7300	3.7300	3.7300	3.6000
v_3	134.0000	32.6400	134.0000	365.0000	134.0000	21.7900
v_4	3.2000	3.2000	2.6000	120.0000	3.2000	2.0000
v_5	32.4000	32.4000	8.7000	32.4000	250.0000	4.5000
v_6	31.6000	31.6000	5.3000	31.6000	280.0000	4.0000



(a) Nonpositive FIF. (b) Rational cubic positive FIF.

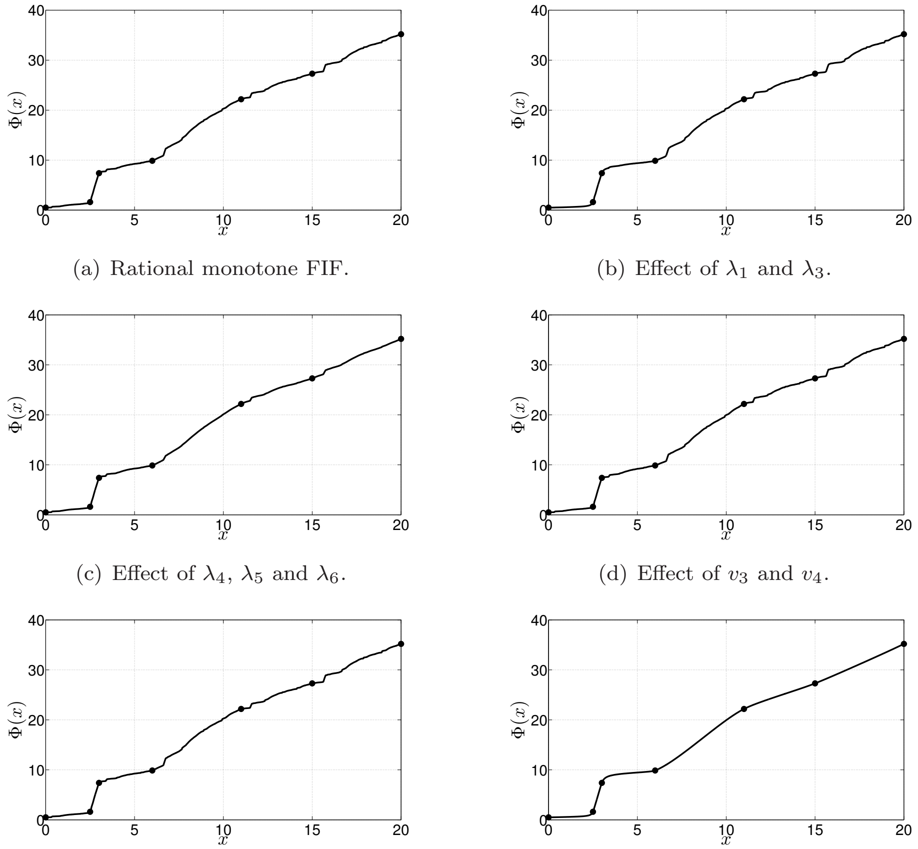
(c) Effect of λ_3 and λ_4. (d) Effect of v_3 and v_4.

(e) Effect of λ_5, λ_6 and λ_7. (f) Classical rational interpolant.

Figure 2: Positivity preserving interpolation.

Table 4: Parameters for convex FIFs with $u_i = 1.5$ for $i = 1, 2, \ldots, 4$.

parameter	Figure 4(a)	Figure 4(b)	Figure 4(c)	Figure 4(d)	Figure 4(e)	Figure 4(f)
λ_1	0.1000	0.0113	0.0113	0.0113	0	0.0430
λ_2	0.2000	0.0434	0.0043	0.0434	0	0.0740
λ_3	0.0100	0.0020	0.0020	0.0020	0	0.0100
λ_4	0.2000	0.0081	0.0081	0.0081	0	0.0160
v_1	2.4000	3.3692	3.3692	3.3692	2.5000	5.6000
v_2	14.0000	16.5544	14.9146	50.0000	18.8158	3.0000
v_3	4.6000	17.1542	17.1542	17.1542	13.9545	6.2000
v_4	3.0000	7.3637	7.3637	7.3637	4.4375	4.0000



(a) Rational monotone FIF.

(b) Effect of λ_1 and λ_3.

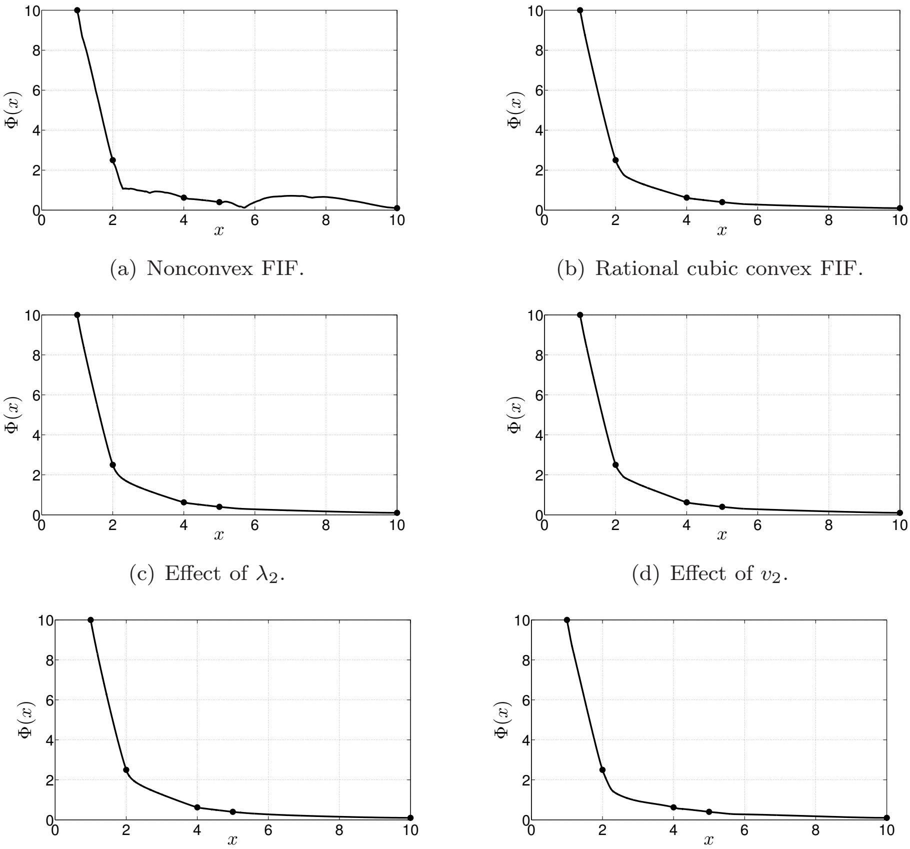
(c) Effect of λ_4, λ_5 and λ_6.

(d) Effect of v_3 and v_4.

(e) Effect of v_5 and v_6.

(f) Classical rational interpolant.

Figure 3: Monotonicity preserving interpolation.



(a) Nonconvex FIF.

(b) Rational cubic convex FIF.

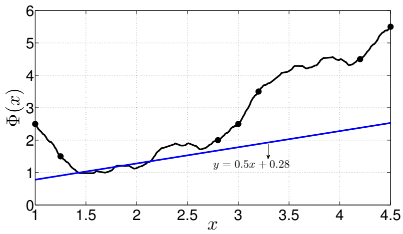
(c) Effect of λ_2.

(d) Effect of v_2.

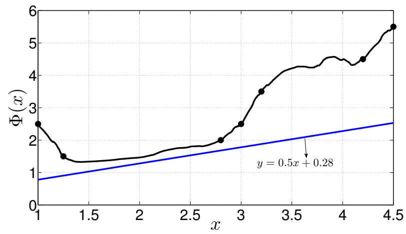
(e) Classical convex interpolant.

(f) Convex interpolant with arbitrary parameters.

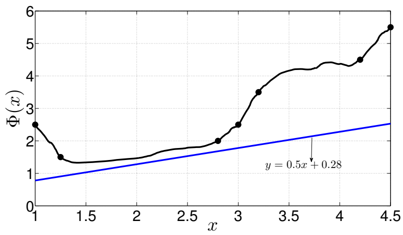
Figure 4: Convexity preserving interpolation.



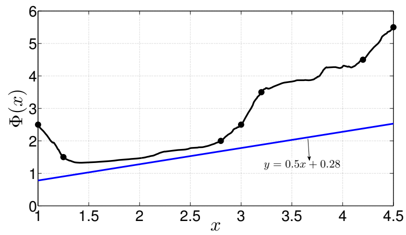
(a) Unconstrained FIF.



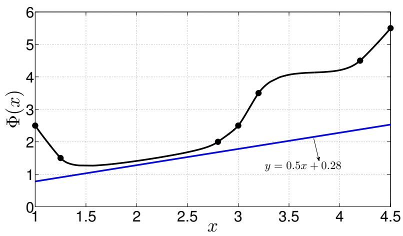
(b) Constrained FIF.



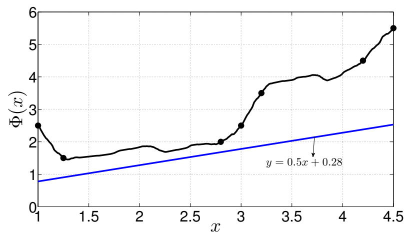
(c) Effect of λ_5.



(d) Effect of v_6.



(e) Classical constrained FIF.



(f) Constrained FIF with arbitrary parameters.

Figure 5: Constrained interpolation.

Table 5: Parameters for constrained FIFs with $u_i = 1.5$ for $i = 1, 2, \ldots, 6$.

parameter	Figure 5(a)	Figure 5(b)	Figure 5(c)	Figure 5(d)	Figure 5(e)	Figure 5(f)
λ_1	0.07	0.07	0.07	0.07	0	-0.06
λ_2	0.4	0.09	0.09	0.09	0	-0.2
λ_3	0.05	0.05	0.05	0.05	0	-0.04
λ_4	0.05	0.05	0.05	0.05	0	0.05
λ_5	0.28	0.28	0.18	0.28	0	-0.18
λ_6	0.08	0.08	0.08	0.08	0	0.08
v_1	3	3	3	3	3	3
v_2	3	46	46	46	14	30
v_3	3	3	3	3	3	3
v_4	3	3	3	3	3	3
v_5	3	3	3	10	3	3
v_6	3	3	3	3	3	3

REFERENCES

[1] M.F. Barnsley, Fractal functions and interpolation, *Constr. Approx.*, 2:303–329, 1986.

[2] M.F. Barnsley, *Fractals Everywhere*, Academic press, Boston, 2014.

[3] M.F. Barnsley and A. N. Harrington, The calculus of fractal interpolation functions, *J. Approx. Theory*, 57:14–34, 1989.

[4] L.D. Berkovitz, *Convexity and Optimization in* $\mathbb{R}^n$, John Wiley and Sons, New York, 2003.

[5] A.K.B. Chand and G.P. Kapoor, Generalized cubic spline fractal interpolation functions, *SIAM J. Numer. Anal.*, 44:655–676, 2006.

[6] A.K.B. Chand and M.A. Navascués, Natural bicubic spline fractal interpolation, *Nonlinear Anal.*, 69:3679–3691, 2008.

[7] A. K. B. Chand and K. R. Tyada, Constrained 2D data interpolation using rational cubic fractal functions, *Springer Proc. Math. Stat., Springer*, 593–607, 2015.

[8] A.K.B. Chand and N. Vijender, Monotonicity preserving rational quadratic fractal interpolation functions, *Adv. Numer. Anal.*, Article ID 504825, 17 pages, 2014.

[9] A.K.B. Chand, N. Vijender and M.A. Navascués, Shape preservation of scientific data through rational fractal splines, *Calcolo*, 51:329–362, 2014.

[10] A. K. B. Chand and P. Viswanathan, A constructive approach to cubic Hermite fractal interpolation function and its constrained aspects, *BIT*, 53:841–865, 2013.

[11] A. K. B. Chand and P. Viswanathan, Shape preserving rational cubic spline fractal interpolation, *arXiv*:1503.02458, 2015.

[12] A.K.B. Chand, P. Viswanathan and K.M. Reddy, Towards a more general type of univariate constrained interpolation with fractal splines, *Fractals*, 23:12 pages, 2015.

[13] R. Delbourgo and J.A. Gregory, $\mathcal{C}^2$ rational quadratic spline interpolation to monotonic data, *IMA J. Numer. Anal.*, 3:141–152, 1983.

[14] R. Delbourgo and J.A. Gregory, Shape preserving piecewise rational interpolation, *SIAM J. Sci. Statist. Comput.*, 6:967–976, 1985.

[15] J.A. Gregory, Shape preserving spline interpolation, *Comput.-Aided Des.*, 18:53–57, 1986.

[16] M.Z. Hussain and M. Hussain, Visualization of data preserving monotonicity, *Appl. Math. Comput.*, 190:1353–1364, 2007.

[17] A.D. Miller and R. Vyborny, Some remarks on functions with one-sided derivatives, *Amer. Math. Monthly*, 471–475, 1986.

[18] M.A. Navascués and M.V. Sebastián, Smooth fractal interpolation, *J. Inequal. Appl.*, Article ID 78734, 1–20, 2006.

[19] M. Sarfraz and M.Z. Hussain, Data visualization using rational spline interpolation, *J. Comput. Appl. Math.*, 189:513–525, 2006.

[20] M. Sarfraz, M.Z. Hussain and M. Hussain, Modeling rational spline for visualization of shaped data, *J. Numer. Math.*, 21:63–88, 2013.

[21] P. Viswanathan and A. K. B. Chand, A fractal procedure for monotonicity preserving interpolation, *Appl. Math. Comput.*, 247:190–204, 2014.

[22] P. Viswanathan and A.K.B. Chand, A $\mathcal{C}^1$-rational cubic fractal interpolation function: convergence and associated parameter identification problem, *Acta Appl. Math.*, 136:19–41, 2015.

[23] P. Viswanathan, A.K.B. Chand and R.P. Agarwal, Preserving convexity through rational cubic spline fractal interpolation function, *J. Comput. Appl. Math.*, 263:262–276, 2014.